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HIGH-FREQUENCY REFRACTION AND DIFFRACTION IN GENERAL MEDIA*

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CONTENTS

The working hypothesis of this paper is that the effect of an opaque boundary on the propagation of high-frequency waves in a general medium is to produce a wave reflected according to the laws of geometrical optics together with a field which to a first approximation depends upon the difference between the curvatures of a tangent ray and the boundary. In order to determine the latter field the model of a medium, whose properties vary linearly, above a straight boundary is employed. A first approximation to the field with this model is found, together with an estimate of the error. The formula for the field is then cast into a form which is invariant under a conformal mapping. Since the difference in curvatures of a tangent ray and the boundary is invariant it is suggested that the field is applicable for all media and boundaries provided that certain conditions imposed in deriving the approximation are fulfilled.

As a check the predictions of the formula are compared with independent calculations on (i) a stratified medium above a straight boundary, (ii) a circular cylinder in a homogeneous medium, (iii) a parabolic cylinder in a homogeneous medium, (iv) a circular cylinder in a circularly stratified medium. In all cases the two calculations are in agreement.

In a final section the results are extended to phenomena which are aperiodic in time.

The proposed universal formula is simple to apply, requiring only the calculation of rays in the medium.

* The substance of this paper was given at the British Theoretical Mechanics Colloquium, 1961.

Introduction

The problem of the propagation of waves in an inhomogeneous medium in the presence of an obstacle arises in a number of different applications, e.g. for electromagnetic, acoustic and elastic waves whenever the properties of the medium vary from point to point. Frequently, too, the medium is anisotropic but in the following it will always be assumed that the medium is isotropic.

Considerable effort has been devoted to this problem in past years and a convenient summary of much of the work is contained in Brekhovskikh's book (1960). Recent contributions to the theory are due to Friedlander (1955), Seckler & Keller (1959) and Felsen (1959).

When the medium is homogeneous the theory is in some respects simpler and contributions have been made by White (1922), Van der Pol & Bremmer (1937, 1938, 1939), Fock (1945, 1946, 1948, 1951), Bremmer (1949), Franz & Depperman (1952), Franz (1954), Rice (1954), Keller (1956), Wu (1956), Kear (1956), Beckmann & Franz (1957), Jeffreys & Lapwood (1957), Jones (1957 *a*, *b*, 1962), Jones & Whitham (1957), Levy & Keller (1957), Goriainov (1958), Levy (1958), Goodrich (1959), Clemmow (1959), Wait & Conda (1959).

The object of the following analysis is to provide a new treatment which unifies all the preceding results and shows how they are related. At the same time there is developed a formula that can be evaluated in a comparatively simple manner and is valid for the whole field subject to certain restrictions.

When a high-frequency wave falls on a boundary a first approximation to the field is provided by geometrical optics. It is well known that this approximation is bad in any shadow region and in the neighbourhood where the transition from shadow to illumination occurs. The basic idea of the following paper is that the effects in these regions are mainly dependent on the difference in curvatures of the boundary and a ray tangent to the boundary. This difference takes its simplest form in two extreme cases: (i) when the boundary is straight so that its curvature is zero, (ii) when the medium is homogeneous so that the curvature of the ray is zero. Most attention in the past has been devoted to problems which come under class (ii). Here, however, we shall adopt (i) as the fundamental model. There is a gain in mathematical simplicity by so doing and, moreover, the connexion between the two classes brings out clearly why creeping waves arise in class (ii).

Since the rays of geometrical optics play a considerable part in our theory § 1 is devoted to an exposition of general ray theory. Explicit calculations for rays in a stratified medium are given in § 2, while the particular results for our model (a source in a medium in which the square of the refractive index varies linearly above a straight boundary) are derived in § 3. Section 4 contains the exact solution for our model. Its approximate evaluation by the method of stationary phase and the comparison with ray theory is made in § 5. An alternative evaluation by means of residues is provided in § 6; this method is valid only in a certain region, usually part of the shadow. Both methods fail near the shadow boundary. The behaviour of this field is obtained, together with an estimate of the error, in §§ 7 to 9. Examination of its form in § 10 reveals that if the formula for the behaviour near the shadow boundary is suitably interpreted it contains all the preceding results. In other words a formula valid for the field everywhere is obtained.

In § 11 this formula is recast into a form which is invariant under a conformal mapping. This has two advantages: (i) if the formula is proved to be valid for any particular problem it will be known to be valid for all problems which can be obtained by conformal mapping, and (ii) it replaces the curvature of the tangent ray of our model by the difference in curvatures of the tangent ray and boundary and therefore provides a field which fits in with our basic idea. It is this invariant formula which it is suggested provides the field everywhere for all media and boundaries provided that certain conditions stated more precisely later on are satisfied. Section 12 gives a corresponding formula for the field on the boundary.

In § 13 an independent calculation is made on a medium with monotonically increasing refractive index above a straight boundary and found to agree with our formula. Section 14 is concerned with the predictions of our formula for various obstacles in a homogeneous medium—they agree with independent calculations made for these problems. A circular cylinder in a radially stratified medium is considered in § 15 and confirmation of results obtained independently for certain cases achieved. The agreement with our formula in all these cases provides strong evidence of its universality although not absolute proof.

The deductions which can be made about fields aperiodic in time are set down in § 16.

An appendix contains certain properties of functions required in the text. References to equations in this appendix are prefixed by the letter A, e.g. (A 1).

1. Geometrical optics

The basic problem is to find solutions of

$$\nabla^2\psi + k^2N^2\psi = 0, \tag{1}$$

where k is a positive real constant and N is a non-negative real function of position. If we regard k as the wave number for propagation in a standard homogeneous medium such as free space then N is the refractive index of the medium under consideration.

Although there is no explicit reference to the time t in (1) we may if we wish think of it as the equation for harmonic waves in a medium with equation

$$\nabla^2\psi - \frac{1}{a^2}\frac{\partial^2\psi}{\partial t^2} = 0. \tag{2}$$

With the time variation $e^{i\omega t}$ understood, $k = \omega/a_0$ and $N = a_0/a$ where $a = a_0$ in the standard homogeneous medium. Then large values of k will correspond to high frequencies or small wavelengths.

We may also think of (1) as the Fourier transform with respect to time of (2). In this case the behaviour of solutions of (1) for large k is related to the behaviour of solutions of (2) immediately behind wave fronts, a point which will be taken up in § 16.

From now on, we shall restrict attention to (1) with k large in some sense to be specified precisely later on. Also only two space dimensions will be used so that (1) reduces to

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + k^2N^2\psi = 0. \tag{3}$$

Partial differential equations involving a large parameter can be tackled by the method of geometrical optics or (as it is sometimes known) the WKBJL method. In this method it is assumed that
$$\psi = e^{-ikL}(\gamma_1 + \gamma_2/k + \gamma_3/k^2 + \dots).$$

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Substituting in (3), taking derivatives term by term and collecting together powers of k we obtain

$$k^2(N^2-\operatorname{grad}^2 L)\,\gamma_1-\mathrm{i}k\{\gamma_1\nabla^2L+2\operatorname{grad}L.\operatorname{grad}\gamma_1+\mathrm{i}\gamma_2(N^2-\operatorname{grad}^2 L)\}$$
$$+\gamma_3(N^2-\operatorname{grad}^2 L)-\mathrm{i}(\gamma_2\nabla^2L+2\operatorname{grad}L.\operatorname{grad}\gamma_2)+\nabla^2\gamma_1+O(1/k)=0. \quad (4)$$

To satisfy this equation for large k we equate to zero the coefficients of the powers of k. From the first two powers

$$\operatorname{grad}^2 L = N^2 \quad (5)$$

and

$$\operatorname{div}(\gamma_1^2\operatorname{grad}L)=0. \quad (6)$$

Usually the approximation is stopped at this stage because of the difficulty of calculating results from the succeeding steps. Accordingly

$$\psi=\gamma_1\,\mathrm{e}^{-\mathrm{i}kL} \quad (7)$$

will be regarded as the solution to (3) in so far as geometrical optics is concerned.

Equation (5) is a partial differential equation for the eikonal L. The curves L = constant are the wave fronts of the propagating disturbance. In many ways it is more convenient to work with the rays rather than the wave fronts. The rays are the orthogonal trajectories of the wave fronts and it follows without difficulty from (5) that they satisfy the ordinary differential equations

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(N\frac{\mathrm{d}x}{\mathrm{d}s}\right)=\frac{\partial N}{\partial x},\quad \frac{\mathrm{d}}{\mathrm{d}s}\left(N\frac{\mathrm{d}y}{\mathrm{d}s}\right)=\frac{\partial N}{\partial y}, \quad (8)$$

where s is the arc length along a ray. Once the rays are known L can be calculated because of (5) and the fact that the rays are the orthogonal trajectories of the wave front. Thus

$$L=\int N\,\mathrm{d}s \quad (9)$$

the integration being along a ray. On account of this formula L is often known as the optical path length of a ray; this terminology will be employed below.

The application of the divergence theorem to (6) in a ray tube shows that

$$\gamma_1^2N\delta\sigma=\text{constant} \quad (10)$$

along a ray tube. Here $\delta\sigma$ is the perpendicular distance between two adjacent rays. Equation (10) or the equivalent form (6) gives the intensity law for the amplitude γ_1 of geometrical optics.

Provided that the approximation is valid the problem of the solution of (3) has been reduced to one of solving the ordinary differential equations (8). Since, in principle, this can always be done (even though a computer may be necessary) it remains only to decide the circumstances under which the process may be expected to be valid.

The approximation (7) consists of treating the field locally as a plane wave in a homogeneous medium. This will be reasonable, unless N is very small, provided that significant variations of γ_1 and L only occur over several wavelengths. It is well known (see, for example, Jones 1962) that a layer of small N in a stratified medium reflects a wave perfectly but provides, in addition, a phase change of $\frac{1}{2}\pi$. We shall therefore put this possibility on one side. The approximation may also fail at a caustic where two adjacent rays meet for then $\delta\sigma$ vanishes in (10) and γ_1 becomes infinite. However, for a region of several wavelengths surrounding the intersection we can treat the medium as homogeneous and use the thorough investigations of Macdonald (1913) on caustics in homogeneous media. Macdonald showed how the field behaves near a caustic and also proved that, for points well away from the intersection, the amplitudes on either side are related by the intensity law provided that a phase advance of $\frac{1}{2}\pi$ is introduced for each crossing of a caustic. By using Macdonald's results it is possible to make suitable modifications to geometrical optics to account for caustics.

There is one further important case in which the geometrical optics approximation may break down and that is in a region in which there is a sharp change in N. Then the gradients of γ_1 and L will be large in general and the significance of the orders of the various terms in (4) will be altered. The theory for this case is in a less satisfactory state than that for the two cases just cited and the reasons why this is so can be seen from the particular model which is studied in some detail in this paper.

Before we turn to this problem, however, some formulae for a stratified medium will be derived.

2. Ray theory for a stratified medium

In a stratified medium in which N is constant on parallel planes the y axis is chosen perpendicular to these planes so that N is a function of y only. Then the first equation of (8) reduces to

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(N\frac{\mathrm{d}x}{\mathrm{d}s}\right) = 0.$$

Hence

$$\frac{\mathrm{d}x}{\mathrm{d}s} = \frac{A}{N}, \tag{11}$$

where A is a constant, which can be taken to be non-negative if we limit consideration to increasing x. Since s is the arc-length of a ray

$$\frac{\mathrm{d}y}{\mathrm{d}s} = \pm\left(1-\frac{A^2}{N^2}\right)^{\frac{1}{2}}, \tag{12}$$

the upper sign corresponding to a rising ray and the lower to a falling ray. Hence the equations of rays passing through (x_0, y_0) are

$$x - x_0 = \pm\int_{y_0}^{y} \frac{A}{(N^2-A^2)^{\frac{1}{2}}}\,\mathrm{d}y. \tag{13}$$

Clearly the rays are real only for $N \geqslant A$. If at some point on the rays N falls to the value A the slope is horizontal there and the ray turns from rising to falling or vice versa.

The optical path length is, from (9),

$$L - L_0 = \pm\int_{y_0}^{y} \frac{N^2}{(N^2-A^2)^{\frac{1}{2}}}\,\mathrm{d}y, \tag{14}$$

where L_0 is the value of L at (x_0, y_0).

With regard to the amplitude let $f(x, y, A) = 0$ be the equation of a ray. Let A have the value $A+\delta A$ on an adjacent ray. Then, if (x, y) and $(x+\delta\sigma\cos\beta,\ y+\delta\sigma\sin\beta)$ are the points on the two rays separated by the perpendicular distance $\delta\sigma$ we have

$$f(x+\delta\sigma\cos\beta,\ y+\delta\sigma\sin\beta,\ A+\delta A) = 0$$

whence

$$\delta\sigma\cos\beta\frac{\partial f}{\partial x} + \delta\sigma\sin\beta\frac{\partial f}{\partial y} + \delta A\frac{\partial f}{\partial A} = 0$$

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to the first order. Here f stands for $f(x, y, A)$. Also

$$\frac{\partial f}{\partial y}\cos\beta-\frac{\partial f}{\partial x}\sin\beta=0.$$

since $\delta\sigma$ is perpendicular to the ray. Hence

$$\left\{\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2\right\}^{\frac{1}{2}}\delta\sigma=\left|\frac{\partial f}{\partial A}\right|\delta A.$$

It follows from (10) that

$$\gamma_1^2=\left\{\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2\right\}^{\frac{1}{2}}K\Big/N\left|\frac{\partial f}{\partial A}\right|\delta A, \tag{15}$$

where K is a constant.

For the rising ray

$$f\equiv x-x_0-\int_{y_0}^{y}\frac{A}{(N^2-A^2)^{\frac{1}{2}}}\,\mathrm{d}y$$

so that
$$\frac{\partial f}{\partial x}=1,\quad \frac{\partial f}{\partial y}=-\frac{A}{(N^2-A^2)^{\frac{1}{2}}},\quad \frac{\partial f}{\partial A}=-\int_{y_0}^{y}\frac{N^2}{(N^2-A^2)^{\frac{3}{2}}}\,\mathrm{d}y.$$

Consequently
$$\gamma_1^2=\frac{K}{(N^2-A^2)^{\frac{1}{2}}\left|\int_{y_0}^{y}\frac{N^2}{(N^2-A^2)^{\frac{3}{2}}}\,\mathrm{d}y\right|\delta A} \tag{16}$$

The formula is exactly the same on a falling ray.

Suppose now that there is a line source at $(0, y_0)$ which would produce a field

$$(\tfrac{1}{2}\pi k)^{\frac{1}{2}}\,\mathrm{e}^{-\frac{1}{4}\pi\mathrm{i}}\,H_0^{(2)}(kN_0 r)$$

in a homogeneous medium where N_0 is the value of N at the source and r is the distance from the source. For large argument the asymptotic approximation

$$H_0^{(2)}(x)\sim\left(\frac{2}{\pi x}\right)^{\frac{1}{2}}\mathrm{e}^{-\mathrm{i}(x-\frac{1}{4}\pi)}$$

for the Hankel function may be employed. Thus a few wavelengths from the source a field $\mathrm{e}^{-\mathrm{i}kN_0r}/(N_0 r)^{\frac{1}{2}}$ is produced. Since we are concerned with high frequencies in ray theory we determine the amplitude so that a field $\mathrm{e}^{-\mathrm{i}kN_0r}/(N_0 r)^{\frac{1}{2}}$ is produced near the source. Consider a ray going down from the source. At a depth $-\mathrm{d}y$ below the source the square of the amplitude is $1/N_0 r$ or $(1-A^2/N_0^2)^{\frac{1}{2}}/N_0\,\mathrm{d}y$ from (12). The difference in abscissae between two adjacent rays at this depth is

$$\frac{A+\delta A}{\{N_0^2-(A+\delta A)^2\}^{\frac{1}{2}}}\,\mathrm{d}y-\frac{A}{(N_0^2-A^2)^{\frac{1}{2}}}\,\mathrm{d}y=\frac{N_0^2\,\delta A\,\mathrm{d}y}{(N_0^2-A^2)^{\frac{3}{2}}}$$

so that
$$\delta\sigma=\frac{N_0^2\,\delta A\,\mathrm{d}y}{(N_0^2-A^2)^{\frac{3}{2}}}\left(-\frac{\mathrm{d}y}{\mathrm{d}s}\right)=\frac{N_0\,\delta A\,\mathrm{d}y}{N_0^2-A^2}.$$

Consequently
$$K=\gamma^2 N_0\,\delta\sigma=\frac{\delta A}{(N_0^2-A^2)^{\frac{1}{2}}}. \tag{17}$$

Substitution in (16) shows that on a ray coming down from the source

$$\gamma_i^2=\frac{1}{(N_0^2-A^2)^{\frac{1}{2}}\,(N^2-A^2)^{\frac{1}{2}}\left|\int_{y_0}^{y}\frac{N^2}{(N^2-A^2)^{\frac{3}{2}}}\,\mathrm{d}y\right|}. \tag{18}$$

The field due to the source is thus $\gamma_i e^{-ikL_i}$ where, from (14),

$$L_i = -\int_{y_0}^{y} \frac{N^2}{(N^2-A^2)^{\frac{1}{2}}}\,dy. \tag{19}$$

If at some level $y = h$ we have $N = A$ (18) must be evaluated as a limit as $y \to h$ to give the amplitude γ_h at the turning point. A straightforward calculation reveals

$$\gamma_h^2 = \frac{N'_h/N_h}{(N_0^2-N_h^2)^{\frac{1}{2}}}, \tag{20}$$

where N_h, N'_h are the values of N and dN/dy at $y = h$. In fact $N_h = A$. The abscissa x_h of the point of turning is given by

$$x_h = -\int_{y_0}^{h} \frac{A}{(N^2-A^2)^{\frac{1}{2}}}\,dy \tag{21}$$

and the ray rising from this point has equation

$$x + \int_{y_0}^{h} \frac{A}{(N^2-A^2)^{\frac{1}{2}}}\,dy = \int_{h}^{y} \frac{A}{(N^2-A^2)^{\frac{1}{2}}}\,dy. \tag{22}$$

In view of the vanishing of N^2-A^2 at $y = h$ it is convenient to integrate by parts before calculating $\partial f/\partial A$ for (15). If this be done and substituted in (15) we find

$$\gamma_i^2 = \frac{1}{(N^2-A^2)^{\frac{1}{2}}\,(N_0^2-A^2)^{\frac{1}{2}}\,|\partial f/\partial A|}, \tag{23}$$

where

$$\left|\frac{\partial f}{\partial A}\right| = \left|\frac{2A^2-N_0^2}{N_0N_0'(N_0^2-A^2)^{\frac{1}{2}}} + \frac{2A^2-N^2}{NN'(N^2-A^2)^{\frac{1}{2}}} + \int_{y_0}^{h} \frac{2A^2-N^2}{(N^2-A^2)^{\frac{1}{2}}}\frac{d}{dy}\left(\frac{1}{NN'}\right)dy \right.$$
$$\left. + \int_{y}^{h} \frac{2A^2-N^2}{(N^2-A^2)^{\frac{1}{2}}}\frac{d}{dy}\left(\frac{1}{NN'}\right)dy\right|.$$

Hence the field on the ray rising from the turning point is $\gamma_i e^{-ikL_i}$ where γ_i is given by (23) and

$$L_i = \int_{h}^{y_0} \frac{N^2}{(N^2-A^2)^{\frac{1}{2}}}\,dy + \int_{h}^{y} \frac{N^2}{(N^2-A^2)^{\frac{1}{2}}}\,dy. \tag{24}$$

Obviously the process can be continued provided that there are no caustics present.

3. The stratified medium with linear variation

Ray theory may be expected to fail when there is a sharp change in N. According to our basic idea the effects should be dependent, to a first approximation, on the difference between the curvatures of a ray and of the sharp change. The model selected for an examination of the phenomenon consists of a medium bounded by a straight boundary. The medium will be assumed to have a refractive index given by

$$N^2 = N_1^2(1+qy), \tag{25}$$

where N_1 and q are positive constants. The sharp change will be represented by an opaque boundary at $y = 0$ on which the boundary condition is

$$\partial\psi/\partial y + kN_1\kappa^{-\frac{1}{3}}Z\psi = 0, \tag{26}$$

where Z is a constant which may be complex and

$$\kappa = kN_1/q.$$

The reason for the insertion of the factor $\kappa^{-\frac{1}{3}}$ will become clearer later.

The equation to be satisfied by ψ in $y > 0$ is

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+k^2N_1^2(1+qy)\,\psi = 0$$

except near the line source at $(0, y_0)$, $(y_0 > 0)$ where a field $e^{-ikN_0r}/(N_0 r)^{\frac{1}{2}}$ is required. Here N_0 is the value of N at the source and so is $N_1(1+qy_0)^{\frac{1}{2}}$.

According to ray theory, a ray coming down from the source will have equation, from (13),

$$x = \frac{2A}{N_1^2 q}\left[\{N_1^2(1+qy_0)-A^2\}^{\frac{1}{2}}-\{N_1^2(1+qy)-A^2\}^{\frac{1}{2}}\right] \tag{27}$$

and will carry a field $\gamma_i e^{-ikL_i}$ where, from (18) and (19),

$$\begin{aligned} L_i = &-\frac{2}{3N_1^2 q}\{N_1^2(1+qy)-A^2\}^{\frac{3}{2}}-\frac{2A^2}{N_1^2 q}\{N_1^2(1+qy)-A^2\}^{\frac{1}{2}} \\ &+\frac{2}{3N_1^2 q}\{N_1^2(1+qy_0)-A^2\}^{\frac{3}{2}}+\frac{2A^2}{N_1^2 q}\{N_1^2(1+qy_0)-A^2\}^{\frac{1}{2}} \\ = &\frac{2}{3N_1^2 q}\left[\{N_1^2(1+qy_0)-A^2\}^{\frac{3}{2}}-\{N_1^2(1+qy)-A^2\}^{\frac{3}{2}}\right]+Ax, \end{aligned} \tag{28}$$

$$\gamma_i^2 = \frac{\frac{1}{2}N_1^2 q}{\{N_1^2(1+qy_0)-2A^2\}\{N_1^2(1+qy)-A^2\}^{\frac{1}{2}}-\{N_1^2(1+qy)-2A^2\}\{N_1^2(1+qy_0)-A^2\}^{\frac{1}{2}}}. \tag{29}$$

If $N_1(1+qy_0)^{\frac{1}{2}} > A > N_1$ the ray turns at a height $(A^2-N_1^2)/N_1^2 q$ above the boundary $y=0$ and subsequently has as its equation

$$x = \frac{2A}{N_1^2 q}\left[\{N_1^2(1+qy_0)-A^2\}^{\frac{1}{2}}+\{N_1^2(1+qy)-A^2\}^{\frac{1}{2}}\right] \tag{30}$$

from (22). It carries a field $\gamma_i e^{-ikL_i}$ where, from (23) and (24),

$$L_i = \frac{2}{3N_1^2 q}\left[\{N_1^2(1+qy_0)-A^2\}^{\frac{3}{2}}+\{N_1^2(1+qy)-A^2\}^{\frac{3}{2}}\right]+Ax, \tag{31}$$

$$\gamma_i^2 = \frac{\frac{1}{2}N_1^2 q}{\left[\{N_1^2(1+qy)-A^2\}^{\frac{1}{2}}+\{N_1^2(1+qy_0)-A^2\}^{\frac{1}{2}}\right]\left|\{N_1^2(1+qy)-A^2\}^{\frac{1}{2}}\{N_1^2(1+qy_0)-A^2\}^{\frac{1}{2}}-A^2\right|}. \tag{32}$$

If $A < N_1$ the downgoing ray strikes the boundary and is reflected. According to ray theory the field behaves locally as a plane wave and so will be reflected by the boundary at an equal angle to the normal with the amplitude multiplied by the Fresnel reflexion coefficient for such a boundary. Since the incident ray makes an χ with the vertical at $y = 0$ where $\sin\chi = A/N_1$ the reflected ray has equation

$$x = \frac{2A}{N_1^2 q}\left[\{N_1^2(1+qy_0)-A^2\}^{\frac{1}{2}}+\{N_1^2(1+qy)-A^2\}^{\frac{1}{2}}-2(N_1^2-A^2)^{\frac{1}{2}}\right] \tag{33}$$

and the appropriate Fresnel reflexion coefficient is

$$R(\chi) = \frac{i\kappa^{\frac{1}{3}}\cos\chi+Z}{i\kappa^{\frac{1}{3}}\cos\chi-Z}. \tag{34}$$

Away from the boundary the reflected ray carries a field $\gamma_r \, \mathrm{e}^{-\mathrm{i}kL_r}$ where

$$\gamma_r = \frac{R(\chi)}{(N^2-A^2)^{\frac{1}{4}}\,(N_0^2-A^2)^{\frac{1}{4}}\,|\partial f_1/\partial A|^{\frac{1}{2}}}, \tag{35}$$

$$L_r = \frac{2}{3N_1^2 q}\left[\{N_1^2(1+qy)-A^2\}^{\frac{3}{2}}+\{N_1^2(1+qy_0)-A^2\}^{\frac{3}{2}}-2(N_1^2-A^2)^{\frac{3}{2}}\right]+Ax \tag{36}$$

and f_1 is the right-hand side of (33).

The ray with $A = N_1$ is tangent to the boundary and has equation

$$\tfrac{1}{2}q^{\frac{1}{2}}x = y^{\frac{1}{2}}+y_0^{\frac{1}{2}}. \tag{37}$$

The mechanism for obtaining the reflected wave clearly fails for rays in the neighbourhood of this one. Moreover, all the reflected rays described above lie to the left of the curve (37)

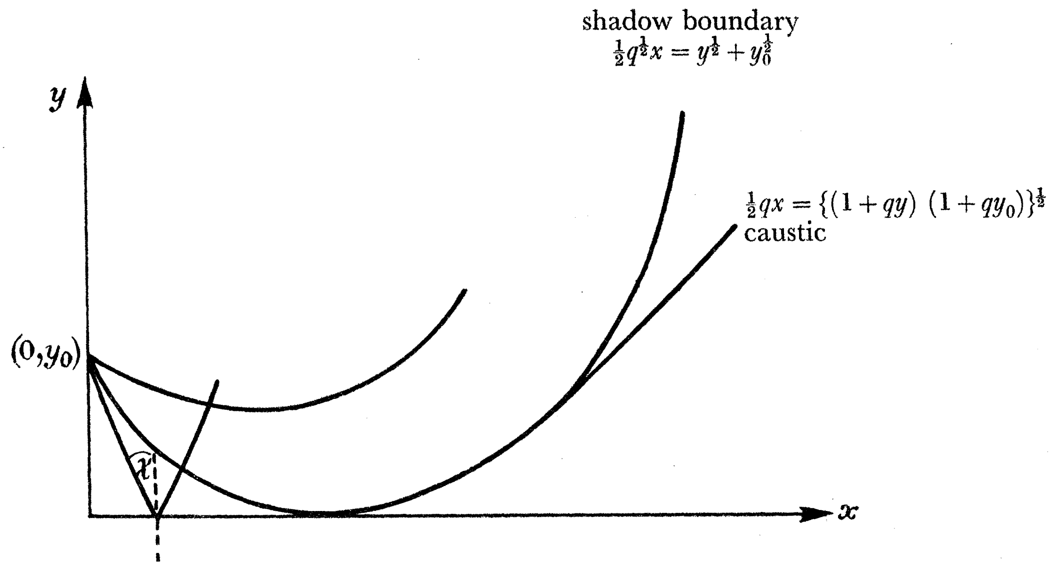

FIGURE 1. The rays from a source at $(0, y_0)$ in a stratified medium with linear variation.

(see figure 1). For this reason the curve (37) is known as the shadow boundary or horizon. There is no means for determining the field by means of rays to the right of the shadow boundary, i.e. in the shadow.

However, it should be noted that if $q^2yy_0 > 1$, some rays can cross the shadow boundary and intersect on a caustic with equation

$$\tfrac{1}{2}qx = \{(1+qy)\,(1+qy_0)\}^{\frac{1}{2}}$$

which touches the shadow boundary at $y = 1/q^2y_0$. Formula (32) is not valid near the caustic; after the ray leaves the caustic (32) again becomes valid but the field must be taken as $\gamma_i \, \mathrm{e}^{-\mathrm{i}kL_i+\frac{1}{2}\pi\mathrm{i}}$.

Hence when an opaque boundary is present, ray theory is inadequate to deal with the field near and beyond the shadow boundary. In the next section the exact solution to the problem is considered; in the course of the investigation confirmation of the formulae of ray theory will be obtained.

4. The exact solution

Let
$$\Psi = \int_{-\infty}^{\infty} \psi \, e^{i\alpha k N_1 x} \, dx.$$

Then
$$d^2\Psi/dy^2 + k^2 N_1^2 (1-\alpha^2+qy)\Psi = 0.$$

The transformation $qy+1-\alpha^2 = Y\kappa^{-\frac{2}{3}}$ converts this to
$$d^2\Psi/dY^2 + Y\Psi = 0$$

which has the solutions $\mathrm{Ai}\,(Y e^{\frac{1}{3}\pi i})$, $\mathrm{Ai}\,(Y e^{-\frac{1}{3}\pi i})$, $\mathrm{Ai}\,(Y e^{\pi i})$ where $\mathrm{Ai}\,(z)$ is the Airy function; it satisfies
$$\mathrm{Ai}''\,(z) = z\,\mathrm{Ai}\,(z). \tag{38}$$

An integral representation is
$$\mathrm{Ai}\,(z) = -\frac{i}{2\pi}\int_{\infty e^{-\frac{1}{3}\pi i}}^{\infty e^{\frac{1}{3}\pi i}} e^{\frac{1}{3}w^3 - zw}\, dw \tag{39}$$

which reduces to
$$\mathrm{Ai}\,(x) = \frac{1}{\pi}\int_0^{\infty} \cos\left(\tfrac{1}{3}t^3 + xt\right) dt \tag{40}$$

for real x. The Airy function is an integral function of z and has the asymptotic behaviour
$$\mathrm{Ai}\,(z) \sim \frac{\exp\left(-\frac{2}{3}z^{\frac{3}{2}}\right)}{2\pi^{\frac{1}{2}} z^{\frac{1}{4}}}\left\{1 + O\left(\frac{1}{|z|}\right)\right\} \quad (|\arg z| < \pi), \tag{41}$$
$$\sim \frac{e^{\frac{1}{4}\pi i}}{\pi^{\frac{1}{2}} z^{\frac{1}{4}}}\left\{\cos\left(\tfrac{2}{3}iz^{\frac{3}{2}} - \tfrac{1}{4}\pi\right) + \exp\left(|\mathscr{R} z^{\frac{3}{2}}|\right) O\left(\frac{1}{|z|}\right)\right\} \quad \left(\tfrac{1}{3}\pi < \arg z < \tfrac{5}{3}\pi\right) \tag{42}$$

as $|z| \to \infty$.

From these asymptotic formulae it is evident that, as $Y\to\infty$, $\mathrm{Ai}\,(Y e^{\pi i})$ represents a standing wave, whereas $\mathrm{Ai}\,(Y e^{\frac{1}{3}\pi i})$ and $\mathrm{Ai}\,(Y e^{-\frac{1}{3}\pi i})$ represent outgoing and incoming waves, respectively. As $Y\to-\infty$, $\mathrm{Ai}\,(Y e^{\pi i})$ is exponentially damped whereas the other solutions increase exponentially. Since the wave due to the line source must not be unbounded at infinity and must produce an outgoing wave as $Y\to\infty$, the incident field due to the line source is given by
$$\Psi_i = B\,\mathrm{Ai}\,\{\kappa^{\frac{2}{3}}(1+qy-\alpha^2)\,e^{\frac{1}{3}\pi i}\}\,\mathrm{Ai}\,\{\kappa^{\frac{2}{3}}(1+qy_0-\alpha^2)\,e^{\pi i}\} \quad (y>y_0)$$
$$= B\,\mathrm{Ai}\,\{\kappa^{\frac{2}{3}}(1+qy_0-\alpha^2)\,e^{\frac{1}{3}\pi i}\}\,\mathrm{Ai}\,\{\kappa^{\frac{2}{3}}(1+qy-\alpha^2)\,e^{\pi i}\} \quad (y<y_0),$$

where
$$B = \frac{4\,.\,2^{\frac{1}{3}}\pi^{\frac{3}{2}}\,e^{\frac{1}{12}\pi i}}{k^{\frac{1}{2}} N_1^{\frac{3}{2}} q^{\frac{1}{3}}}.$$

The scattered field produced by the boundary $y=0$ can be accounted for by adding a suitable constant multiple of $\mathrm{Ai}\,\{\kappa^{\frac{2}{3}}(1+qy-\alpha^2)\,e^{\frac{1}{3}\pi i}\}$. An inverse Fourier transform then gives ψ.

The resultant formula is, for $y<y_0$,
$$\psi = \frac{BkN_1}{2\pi}\int_{-\infty}^{\infty} \mathrm{Ai}\,\{\kappa^{\frac{2}{3}}(1+qy_0-\alpha^2)\,e^{\frac{1}{3}\pi i}\}\,[\mathrm{Ai}\,\{\kappa^{\frac{2}{3}}(1+qy-\alpha^2)\,e^{\pi i}\}$$
$$+ R_1(\alpha^2)\,\mathrm{Ai}\,\{\kappa^{\frac{2}{3}}(1+qy-\alpha^2)\,e^{\frac{1}{3}\pi i}\}]\,e^{-i\alpha k N_1 x}\,d\alpha, \tag{43}$$

where
$$R_1(\alpha^2) = -\frac{e^{\pi i}\,\mathrm{Ai}'\,\{\kappa^{\frac{2}{3}}(1-\alpha^2)\,e^{\pi i}\} + Z\,\mathrm{Ai}\,\{\kappa^{\frac{2}{3}}(1-\alpha^2)\,e^{\pi i}\}}{e^{\frac{1}{3}\pi i}\,\mathrm{Ai}'\,\{\kappa^{\frac{2}{3}}(1-\alpha^2)\,e^{\frac{1}{3}\pi i}\} + Z\,\mathrm{Ai}\,\{\kappa^{\frac{2}{3}}(1-\alpha^2)\,e^{\frac{1}{3}\pi i}\}}. \tag{44}$$

It is assumed that the denominator of R_1 does not vanish on the real axis.

From now on it will be assumed that $y \leqslant y_0$. There is no loss of generality in making this assumption because the reciprocity theorem asserts that the field at y is the same as the field at y_0 due to a line source at y. It will also be assumed that $x \geqslant 0$.

Although (43) is the exact solution to the problem for all k it is scarcely in a form which is amenable to calculation. It is, however, suitable for calculating the high-frequency behaviour. The condition for high frequencies is

$$kN_1/q \gg 1. \tag{45}$$

5. The method of stationary phase

One method of evaluating an integral involving a large parameter is that of stationary phase. In this section we shall consider the application of that method to (43).

The integral from $-\infty$ to 0 can be replaced by one from 0 to ∞ with x replaced by $-x$ so that for the moment discussion of the integrand will be limited to $\alpha \geqslant 0$. Since kN_1/q is large it is reasonable to replace each Airy function by its asymptotic formula except when α^2 is near 1, $1+qy$ or $1+qy_0$. If each Airy function is so replaced it is evident that an exponential is obtained which has the large factor kN_1/q in the exponent so that the method of stationary phase can be applied. In the range $\alpha^2 > 1+qy$ the first term of the integrand is exponentially decreasing so that there is no point of stationary phase in this range. In $1 < \alpha^2 < 1+qy$ the equations for the points of stationary phase of the first term are

$$\frac{\partial}{\partial \alpha}\{-\tfrac{2}{3}(1+qy_0-\alpha^2)^{\frac{3}{2}} \pm \tfrac{2}{3}(1+qy-\alpha^2)^{\frac{3}{2}} - \alpha qx\} = 0,$$

or

$$2\alpha(1+qy_0-\alpha^2)^{\frac{1}{2}} \mp 2\alpha(1+qy-\alpha^2)^{\frac{1}{2}} - qx = 0. \tag{46}$$

The two signs have to be considered because of (42). With the upper sign there is one root of (46) provided that $0 < q^{\frac{1}{2}}x < 2(1+qy)^{\frac{1}{2}}(y_0-y)^{\frac{1}{2}}$. This corresponds to the downgoing incident ray before it turns, (46) being the same as (27) when α is replaced by A/N_1, and the contribution from the point of stationary phase is found to be $\gamma_i\,\mathrm{e}^{-\mathrm{i}kL_i}$ where γ_i and L_i are given by (29) and (28).

With the lower sign of (46) there is one root provided that $q^{\frac{1}{2}}x > 2(1+qy)^{\frac{1}{2}}(y_0-y)^{\frac{1}{2}}$ and $qy_0 < 1$. This corresponds to the incident ray rising from its turning point, (46) being the same as (30) if α is replaced by A/N_1, and the contribution from the point of stationary phase gives $\gamma_i\,\mathrm{e}^{-\mathrm{i}kL_i}$ where γ_i and L_i are given by (32) and (31). However, as the point of observation approaches the shadow boundary from the illuminated region the point of stationary phase approaches $\alpha = 1$. Since $\alpha = 1$ is the lower limit of integration the standard method of dealing with a point of stationary phase is no longer applicable and another process must be adopted. This point will be returned to later on.

When $qy_0 > 1$ the situation is unaltered if $y < 1/q^2y_0$. If, however, $q^2yy_0 > 1$ there is one root of (46) with the lower sign when $(1+qy)^{\frac{1}{2}}(y_0-y)^{\frac{1}{2}} < \frac{1}{2}q^{\frac{1}{2}}x < y^{\frac{1}{2}}+y_0^{\frac{1}{2}}$ (this gives the incident wave above) and two roots when $y^{\frac{1}{2}}+y_0^{\frac{1}{2}} < \frac{1}{2}q^{\frac{1}{2}}x < \{(1+qy)(1+qy_0)\}^{\frac{1}{2}}/q^{\frac{1}{2}}$. The greater of these roots gives the direct incident ray as above and is not near $\alpha = 1$ except possibly for $q^2yy_0 \approx 1$. The smaller gives a ray which has passed through the caustic and reproduces $\gamma_i\,\mathrm{e}^{-\mathrm{i}kL_i+\frac{1}{2}\pi\mathrm{i}}$, where γ_i and L_i are given by (32) and (31). As the point of observation approaches the caustic the two roots come together and the third derivative approximation in the method of stationary phase must be employed. This leads to an Airy function expression for the behaviour near a caustic as in Macdonald (1913). As the point of observation approaches the shadow boundary the lower root tends to $\alpha = 1$, and once again the range of integration for the method of stationary phase is incomplete.

In other words, if $qy_0 > 1$, any point in the region to the right of the shadow boundary and to the left of the caustic receives two incident rays, one direct from the source and one via the caustic. The direct rays go straight across the shadow boundary. For the rays from the caustic the shadow boundary forms a natural boundary on the left.

In the range $\alpha^2 \geqslant 1$ the second term of (43) has no real point of stationary phase.

In the range $\alpha^2 \leqslant 1$ use the relation

$$\mathrm{Ai}\,(z\,\mathrm{e}^{\pi\mathrm{i}}) = \mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}\,\mathrm{Ai}\,(z\,\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}) + \mathrm{e}^{-\frac{1}{3}\pi\mathrm{i}}\,\mathrm{Ai}\,(z\,\mathrm{e}^{-\frac{1}{3}\pi\mathrm{i}}) \tag{47}$$

with the net result in (41) that $\mathrm{Ai}\,(z\,\mathrm{e}^{\pi\mathrm{i}})$ and $\mathrm{e}^{\pi\mathrm{i}}\,\mathrm{Ai}'\,(z\,\mathrm{e}^{\pi\mathrm{i}})$ are replaced by $\mathrm{e}^{-\frac{1}{3}\pi\mathrm{i}}\,\mathrm{Ai}(z\,\mathrm{e}^{-\frac{1}{3}\pi\mathrm{i}})$ and $\mathrm{e}^{-\frac{2}{3}\pi\mathrm{i}}\,\mathrm{Ai}'\,(z\,\mathrm{e}^{-\frac{1}{3}\pi\mathrm{i}})$, respectively. With (43) so modified the first term has one real point of stationary phase for $\alpha^2 \leqslant 1$ which provides a downgoing incident ray which strikes the boundary. The second term has one which satisfies

$$\frac{\partial}{\partial\alpha}\{-\tfrac{2}{3}(1+qy_0-\alpha^2)^{\frac{3}{2}} - \tfrac{2}{3}(1+qy-\alpha^2)^{\frac{3}{2}} + \tfrac{4}{3}(1-\alpha^2)^{\frac{3}{2}} - \alpha qx\} = 0,$$

or

$$2\alpha(1+qy_0-\alpha^2)^{\frac{1}{2}} + 2\alpha(1+qy-\alpha^2)^{\frac{1}{2}} - 4\alpha(1-\alpha^2)^{\frac{1}{2}} - qx = 0 \tag{48}$$

since

$$\mathrm{Ai}'\,(z) \sim -\frac{z^{\frac{1}{4}}\exp\,(-\frac{2}{3}z^{\frac{3}{2}})}{2\pi^{\frac{1}{2}}}\left\{1+O\left(\frac{1}{|z|}\right)\right\} \quad (|\arg z| < \pi) \tag{49}$$

$$\sim \frac{z^{\frac{1}{4}}\,\mathrm{e}^{-\frac{1}{4}\pi\mathrm{i}}}{\pi^{\frac{1}{2}}}\left\{\sin\left(\frac{2\mathrm{i}}{3}z^{\frac{3}{2}} - \tfrac{1}{4}\pi\right) + \exp\,(|\mathscr{R}z^{\frac{3}{2}}|)\,O\left(\frac{1}{|z|}\right)\right\} \quad \left(\tfrac{1}{3}\pi < \arg z < \frac{5\pi}{3}\right). \tag{50}$$

Equation (48) becomes the same as (33) on putting $\alpha = A/N_1$ and so this point of stationary phase supplies the ray reflected from the boundary and reproduces (35). As the point of observation approaches the shadow boundary the point of stationary phase approaches $\alpha = 1$ and the process breaks down.

Consequently, the replacement of the Airy functions by their asymptotic forms and the use of the method of stationary phase reproduces the formulae of geometrical optics plus details of the behaviour near the caustic when it is present. No information is obtained about the field in the shadow or near the shadow boundary. Therefore alternative methods of evaluation must be used in these regions.

6. Evaluation by residues

Since the Airy function is an integral function the only singularities of the integrand of (43) are simple poles where

$$\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}\,\mathrm{Ai}'\,\{\kappa^{\frac{2}{3}}(1-\alpha^2)\,\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}\} + Z\,\mathrm{Ai}\,\{\kappa^{\frac{2}{3}}(1-\alpha^2)\,\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}\} = 0. \tag{51}$$

Let δ_s $(s = 1, 2, \ldots)$ satisfy

$$\mathrm{Ai}'\,(\delta_s) + Z\,\mathrm{e}^{-\frac{1}{3}\pi\mathrm{i}}\,\mathrm{Ai}\,(\delta_s) = 0; \tag{52}$$

then the roots of (51) are given by $\alpha = \eta_s$ where

$$\eta_s^2 = 1 - \delta_s\,\mathrm{e}^{-\frac{1}{3}\pi\mathrm{i}}/\kappa^{\frac{2}{3}}. \tag{53}$$

One reason for the insertion of $\kappa^{-\frac{1}{3}}$ in (26) is to obtain the comparatively simple form of (52).

The roots of (52) have been discussed in some detail by van der Pol & Bremmer (1939), Fock (1946) and Franz & Beckmann (1956). If Z is large, $\delta_s \approx \alpha_s e^{\pi i}$ where

$$\mathrm{Ai}\,(\alpha_s e^{\pi i}) = 0. \tag{54}$$

If Z is very small $\delta_s \approx \beta_s e^{\pi i}$ where $$\mathrm{Ai}'\,(\beta_s e^{\pi i}) = 0. \tag{55}$$

Both α_s and β_s are positive and lie between 1 and 10 for $1 \leqslant s \leqslant 6$. Therefore, from (53), for these extreme values of Z there is one set of poles starting near 1 and going along the direction $e^{-\frac{1}{3}\pi i}$. There is a corresponding set in the second quadrant. We assume that the behaviour is similar for intermediate values of Z. For large values of s (49), (50), (41) and (42) show that the derivative term in (52) dominates so that $\delta_s \sim \beta_s e^{\pi i}$ for large s and the corresponding poles are asymptotic to $\pm \beta_s^{\frac{1}{2}} e^{-\frac{1}{6}\pi i} \kappa^{-\frac{1}{3}}$.

Deform the contour of (43) into a large semi-circle in the lower half-plane together with a loop round the poles. The contribution of the semicircle tends to zero as the radius increases provided that

$$|x| > 3^{\frac{1}{2}}(y+y_0). \tag{56}$$

It is helpful to use the modified form of the integrand described in the preceding section just before (48) on the semicircle where $-\pi \leqslant \arg \alpha \leqslant -\frac{1}{6}\pi$.

Hence

$$\psi = \frac{BkN_1 i}{2\kappa^{\frac{2}{3}}} \sum_{s=1} \frac{e^{\pi i}\,\mathrm{Ai}'\,(\delta_s e^{\frac{2}{3}\pi i}) + Z\,\mathrm{Ai}\,(\delta_s e^{\frac{2}{3}\pi i})}{\eta_s e^{\frac{2}{3}\pi i}\mathrm{Ai}''\,(\delta_s) + Z\,\mathrm{Ai}'\,(\delta_s)\, e^{\frac{1}{3}\pi i}\eta_s}$$
$$\times \mathrm{Ai}\,\{\kappa^{\frac{2}{3}}(1+qy_0-\eta_s^2)\, e^{\frac{1}{3}\pi i}\}\,\mathrm{Ai}\,\{\kappa^{\frac{2}{3}}(1+qy-\eta_s^2)\, e^{\frac{1}{3}\pi i}\}\, e^{-i\eta_s kN_1 x}$$

when (56) is satisfied. This expression can be simplified somewhat by using (38), (52) and the Wronskian

$$e^{\frac{2}{3}\pi i}\,\mathrm{Ai}'\,(z e^{\frac{1}{3}\pi i})\,\mathrm{Ai}\,(z e^{\pi i}) - e^{\frac{4}{3}\pi i}\mathrm{Ai}\,(z e^{\frac{1}{3}\pi i})\,\mathrm{Ai}'\,(z e^{\pi i}) = -i/2\pi. \tag{57}$$

The result is

$$\psi = \frac{(2\pi k)^{\frac{1}{2}}}{\kappa^{\frac{1}{3}}} e^{\frac{1}{12}\pi i} \sum_s \frac{Z^2\,\mathrm{Ai}\,\{\kappa^{\frac{2}{3}}(1+qy_0-\eta_s^2)\, e^{\frac{1}{3}\pi i}\}\,\mathrm{Ai}\,\{\kappa^{\frac{2}{3}}(1+qy-\eta_s^2)\, e^{\frac{1}{3}\pi i}\}}{\{Z^2-\delta_s e^{\frac{2}{3}\pi i}\}\{\mathrm{Ai}'\,(\delta_s)\}^2} e^{-i\eta_s kN_1 x}. \tag{58}$$

If $Z = 0$ replace $Z/\mathrm{Ai}'\,(\delta_s)$ by $-e^{\frac{1}{3}\pi i}/\mathrm{Ai}\,(\delta_s)$.

It should be noted that (58) is valid *without restriction on frequency* but subject to (56). The region of validity does not include the whole shadow zone but may include part of the illuminated region. A possible case is illustrated in figure 2. Far enough into the shadow (58) will certainly hold.

Formula (58) could also have been obtained direct from the differential equation for ψ by the method of separation of variables. The fact that it is valid only when (56) is true indicates that separation of variables must be used with circumspection in inhomogeneous media (see also Marcuvitz 1951).

For the early values of s η_s is of the order of unity and the arguments of the Airy functions in (58) are large unless y or y_0 is small. The Airy functions may then be replaced by their asymptotic forms. This will certainly be true if

$$qy \gg |\delta_s|\,(q/kN_1)^{\frac{2}{3}} \tag{59}$$

for the smaller values of s.

Then

$$\psi = \left(\frac{k}{8\pi}\right)^{\frac{1}{2}} \left(\frac{1}{\kappa}\right)^{\frac{2}{3}} \frac{e^{-\frac{1}{12}\pi i}}{(q^2 yy_0)^{\frac{1}{4}}} \sum_{s=1} \frac{Z^2 \exp\{-ikL + \frac{1}{2}i\kappa^{\frac{1}{3}}\delta_s e^{-\frac{1}{3}\pi i}\, qL_b/N_1\}}{\{Z^2-\delta_s e^{\frac{2}{3}\pi i}\}\{\mathrm{Ai}'\,(\delta_s)\}^2}, \tag{60}$$

where
$$L = N_1\{x+\tfrac{2}{3}q^{\frac{1}{2}}(y_0^{\frac{3}{2}}+y^{\frac{3}{2}})\}, \tag{61}$$
$$L_b = N_1\{x-2q^{-\frac{1}{2}}(y^{\frac{1}{2}}+y_0^{\frac{1}{2}})\}. \tag{62}$$

The procedure for going from (58) to (60) is not, of course, valid because, for fixed qy, the inequality (59) is bound to be violated when s is large enough. Indeed the infinite series (60) converges exponentially, at any rate when Z is zero or infinite, when $L_b > 0$, i.e. it converges in whole shadow, subject to (59), except near the shadow boundary $L_b = 0$. Well into the shadow (58) and (60) are in agreement in the sense that only a few terms of either series are necessary there and the approximation made above is valid for these terms. We could attempt to prove (60) by showing that the infinite semicircle used in deriving (58) provides an infinite contribution, where (60) converges but not (58), which cancels the infinity in (58) to provide (60) but we shall not do so.

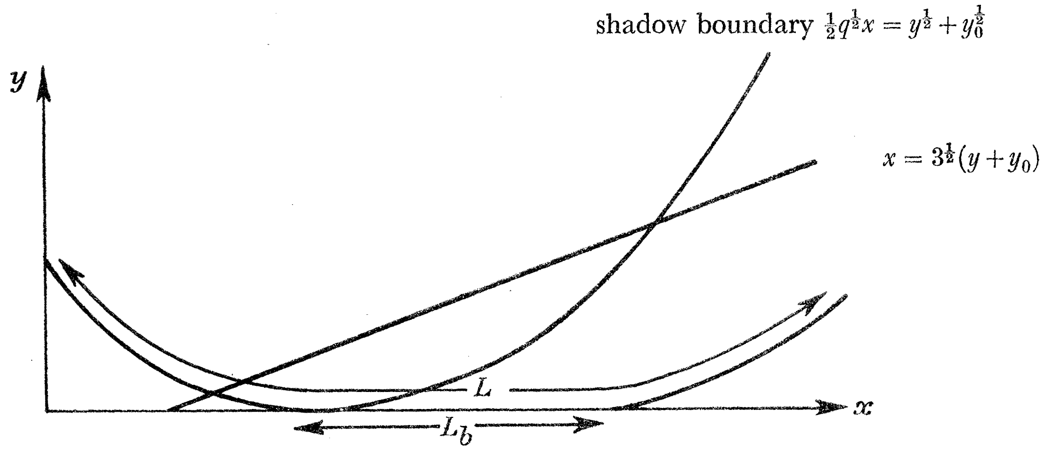

FIGURE 2. The method of residues gives the field in the shadow to the right of the straight line. The exponential decay depends upon L_b, the distance travelled along the boundary.

Formula (60) forms the basis of Seckler's & Keller's (1959) theory of diffraction. In this theory a point in the shadow is assumed to be reached by a ray which consists of the incident downgoing ray which touches the boundary, the upgoing ray which touches the boundary and passes through the point of observation, and the portion of the boundary between the two points of contact. It may be verified from the analysis of § 3 that L is the optical path length of such a ray, L_b being the optical path length of the boundary portion. The occurrence of L in (60) accounts for the normal phase variation along a ray. With regard to L_b it is assumed that a fixed amount of energy is brought along the incident part of the ray. It steadily diminishes in travel along the boundary because energy is carried upwards by each of the tangent upgoing rays. Therefore the amplitude at the point of observation is exponentially decreased by an amount proportional to the length of boundary which has to be covered.

If (60) be accepted the results of this and the preceding section ensure that the field is known everywhere except in the neighbourhood of the shadow boundary. The problem of the shadow boundary will be considered in later sections.

Before turning to this we note that, for
$$qy_0 \gg |\delta_s|\,(q/kN_1)^{\frac{2}{3}} \tag{63}$$

the field on the boundary $y = 0$ according to (58) is

$$\psi = \left(\frac{1}{2}\frac{k}{\kappa}\right)^{\frac{1}{2}} \frac{1}{(qy_0)^{\frac{1}{4}}} \sum_s \frac{Z^2 \exp\{-\mathrm{i}kL + \frac{1}{2}\mathrm{i}\kappa^{\frac{1}{3}}\delta_s\, \mathrm{e}^{-\frac{1}{3}\pi\mathrm{i}} qL_b/N_1\}\, \mathrm{Ai}\,(\delta_s)}{(Z^2 - \delta_s\, \mathrm{e}^{\frac{2}{3}\pi\mathrm{i}})\, \{\mathrm{Ai}'\,(\delta_s)\}^2}, \tag{64}$$

$$\frac{\partial\psi}{\partial y} = (\tfrac{1}{2}k)^{\frac{1}{2}}\, \kappa^{\frac{1}{6}}\, \frac{q\, \mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}}{(qy_0)^{\frac{1}{4}}} \sum_s \frac{Z^2 \exp\{-\mathrm{i}kL + \frac{1}{2}\mathrm{i}\kappa^{\frac{1}{3}}\delta_s\, \mathrm{e}^{-\frac{1}{3}\pi\mathrm{i}} qL_b/N_1\}}{(Z^2 - \delta_s\, \mathrm{e}^{\frac{2}{3}\pi\mathrm{i}})\, \mathrm{Ai}'\,(\delta_s)} \tag{65}$$

which satisfies the boundary condition (26) when account is taken of (52).

7. Extraction of the exponentially damped part of the field

The first step in finding the field near the shadow boundary is to isolate the most significant parts of (43). In the first place we shall split off the parts which are exponentially small at high frequencies whether the point of observation is near the shadow boundary or not. It will, however, be assumed that (59) is valid in the form

$$qy \gg (q/kN_1)^{\frac{2}{3}}. \tag{66}$$

On account of (41) and the Airy function being an integral function there is a K such that

$$|\mathrm{Ai}\,(x)| \leqslant \frac{K \exp\,(-\frac{2}{3}x^{\frac{3}{2}})}{1 + x^{\frac{1}{4}}} \quad (x \geqslant 0), \tag{67}$$

$$|\mathrm{Ai}\,(x\, \mathrm{e}^{\frac{1}{3}\pi\mathrm{i}})| \leqslant \frac{K}{1 + x^{\frac{1}{4}}} \quad (x \geqslant 0), \tag{68}$$

$$|\mathrm{Ai}\,(x\, \mathrm{e}^{\frac{1}{3}\pi\mathrm{i}})| \leqslant \frac{K \exp\,(\frac{2}{3}\,|x|^{\frac{3}{2}})}{1 + |x|^{\frac{1}{4}}} \quad (x \leqslant 0). \tag{69}$$

Because of (67) and (69)

$$\begin{aligned} &\left| \int_{(1+qy_0)^{\frac{1}{2}}}^{\infty} \mathrm{Ai}\,\{\kappa^{\frac{2}{3}}(1 + qy_0 - \alpha^2)\, \mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}\}\, \mathrm{Ai}\,\{\kappa^{\frac{2}{3}}(1 + qy - \alpha^2)\, \mathrm{e}^{\pi\mathrm{i}}\}\, \mathrm{e}^{-\mathrm{i}\alpha kN_1 x}\, \mathrm{d}\alpha \right| \\ &\quad \leqslant K^2 \int_{(1+qy_0)^{\frac{1}{2}}}^{\infty} \exp\,[\tfrac{2}{3}\kappa\{(\alpha^2 - 1 - qy_0)^{\frac{3}{2}} - (\alpha^2 - 1 - qy)^{\frac{3}{2}}\}]\, \mathrm{d}\alpha \\ &\quad \leqslant K^2 \exp\{-\tfrac{2}{3}kN_1 q^{\frac{1}{2}}(y_0 - y)^{\frac{3}{2}}\} \int_0^{\infty} \exp\{-\tfrac{1}{3}kN_1 \alpha(y_0 - y)\}\, \mathrm{d}\alpha \\ &\quad = O[\exp\{-\tfrac{2}{3}kN_1 q^{\frac{1}{2}}(y_0 - y)^{\frac{3}{2}}\}] \end{aligned}$$

since $y_0 > y$. Let $\epsilon = (kN_1/q)^{-\frac{2}{3}+\eta}$ where $\eta > 0$. Then, from (67) and (68),

$$\begin{aligned} &\left| \int_{(1+qy)^{\frac{1}{2}}(1+\epsilon)^{\frac{1}{2}}}^{(1+qy_0)^{\frac{1}{2}}} \mathrm{Ai}\,\{\kappa^{\frac{2}{3}}(1 + qy_0 - \alpha^2)\, \mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}\}\, \mathrm{Ai}\,\{\kappa^{\frac{2}{3}}(1 + qy - \alpha^2)\, \mathrm{e}^{\pi\mathrm{i}}\}\, \mathrm{e}^{-\mathrm{i}\alpha kN_1 x}\, \mathrm{d}\alpha \right| \\ &\quad \leqslant K^2 \int_{(1+qy)^{\frac{1}{2}}(1+\epsilon)^{\frac{1}{2}}}^{(1+qy_0)^{\frac{1}{2}}} \exp\{-\tfrac{2}{3}\kappa(\alpha^2 - 1 - qy)^{\frac{3}{2}}\}\, \mathrm{d}\alpha \\ &\quad = O[\exp\{-\kappa\epsilon^{\frac{3}{2}}\}] \\ &\quad = O\{\exp\,(-\kappa^{\frac{3}{2}\eta})\}. \end{aligned}$$

On account of (49) there is a constant K_1 such that

$$|\mathrm{Ai}'\,(x)| \leqslant K_1(1 + x^{\frac{1}{4}}) \exp\,(-\tfrac{2}{3}x^{\frac{3}{2}}) \quad (x \geqslant 0).$$

Since it has been assumed that the denominator of the second term in (43) does not vanish on the real axis there is a constant $K_2 > 0$ such that

$$|\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}\,\mathrm{Ai}'\,(x\,\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}})+Z\,\mathrm{Ai}\,(x\,\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}})| \geqslant K_2 \exp\left(\tfrac{2}{3}\,|x|^{\frac{3}{2}}\right) \quad (x \leqslant 0).$$

Consequently

$$\int_{(1+qy_0)^{\frac{1}{2}}}^{\infty} \frac{\mathrm{e}^{\pi\mathrm{i}}\,\mathrm{Ai}'\,\{\kappa^{\frac{2}{3}}(1-\alpha^2)\,\mathrm{e}^{\pi\mathrm{i}}\}+Z\,\mathrm{Ai}\,\{\kappa^{\frac{2}{3}}(1-\alpha^2)\,\mathrm{e}^{\pi\mathrm{i}}\}}{\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}\,\mathrm{Ai}'\,\{\kappa^{\frac{2}{3}}(1-\alpha^2)\,\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}\}+Z\,\mathrm{Ai}\,\{\kappa^{\frac{2}{3}}(1-\alpha^2)\,\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}\}}\,\mathrm{Ai}\,\{\kappa^{\frac{2}{3}}(1+qy-\alpha^2)\,\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}\}\,\mathrm{e}^{-\mathrm{i}\alpha kN_1x}\,\mathrm{d}\alpha$$
$$= O[\exp\left(\tfrac{2}{3}kN_1\,q^{\frac{1}{2}}\{(y_0-y)^{\frac{3}{2}}-2y_0^{\frac{3}{2}}\}\right)].$$

The integral over the interval from $(1+qy)^{\frac{1}{2}}$ to $(1+qy_0)^{\frac{1}{2}}$ is $O\{\exp\left(-\tfrac{4}{3}kN_1\,q^{\frac{1}{2}}\,y^{\frac{3}{2}}\right)\}$ and the integral over the interval from $1+\epsilon$ to $(1+qy_0)^{\frac{1}{2}}$ is $O\{\exp\left(-\kappa^{\frac{3}{2}\eta}\right)\}$.

If x is replaced by $-x$ the above results are not altered. Therefore, if exponentially small terms are neglected,

$$\psi = \frac{BkN_1}{2\pi}\int_{-(1+qy)^{\frac{1}{2}}(1+\epsilon)^{\frac{1}{2}}}^{(1+qy)^{\frac{1}{2}}(1+\epsilon)^{\frac{1}{2}}} \mathrm{Ai}\,\{\kappa^{\frac{2}{3}}(1+qy_0-\alpha^2)\,\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}\}\,\mathrm{Ai}\,\{\kappa^{\frac{2}{3}}(1+qy-\alpha^2)\,\mathrm{e}^{\pi\mathrm{i}}\}\,\mathrm{e}^{-\mathrm{i}\alpha kN_1x}\,\mathrm{d}\alpha$$
$$+\frac{BkN_1}{2\pi}\int_{-(1+\epsilon)^{\frac{1}{2}}}^{(1+\epsilon)^{\frac{1}{2}}} R_1(\alpha^2)\,\mathrm{Ai}\,\{\kappa^{\frac{2}{3}}(1+qy_0-\alpha^2)\,\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}\}\,\mathrm{Ai}\,\{\kappa^{\frac{2}{3}}(1+qy-\alpha^2)\,\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}\}\,\mathrm{e}^{-\mathrm{i}\alpha kN_1x}\,\mathrm{d}\alpha. \quad (70)$$

This formula is valid when the conditions (45) and (66) hold. The correction term is conveniently written $O\{\exp\left(-\kappa^{\frac{3}{2}\eta}\right)\}$ on imposing the restrictions

$$q(y_0-y) > 2\epsilon, \quad qy > \epsilon. \quad (71)$$

8. The incident field term near the shadow boundary

We turn now to a more detailed discussion of the integrals in (70). In order that the examination shall be as general as possible restriction to the neighbourhood of the shadow boundary will be avoided as far as possible. However, near the shadow boundary two simplifications are helpful, namely, that $L_b \approx 0$ and $L \approx L_i$ as can be seen from (31) with $A \approx N_1$. This section will be concerned only with the first term of (70).

Now

$$\int_{1}^{(1+qy)^{\frac{1}{2}}(1+\epsilon)^{\frac{1}{2}}} \mathrm{Ai}\,\{\kappa^{\frac{2}{3}}(1+qy_0-\alpha^2)\,\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}\}\,\mathrm{Ai}\,\{\kappa^{\frac{2}{3}}(1+qy-\alpha^2)\,\mathrm{e}^{\pi\mathrm{i}}\}\,\mathrm{e}^{-\mathrm{i}\alpha kN_1x}\,\mathrm{d}\alpha$$
$$= \int_{1}^{(1+qy)^{\frac{1}{2}}(1+\epsilon)^{\frac{1}{2}}} \frac{\exp\{-\frac{2}{3}\mathrm{i}\kappa(1+qy_0-\alpha^2)^{\frac{3}{2}}\}}{2\pi^{\frac{1}{2}}\,\kappa^{\frac{1}{6}}(1+qy_0-\alpha^2)^{\frac{1}{4}}\,\mathrm{e}^{\frac{1}{12}\pi\mathrm{i}}}\left\{1+O\left(\frac{1}{\kappa^{\frac{2}{3}}(1+qy_0-\alpha^2)\,\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}}\right)\right\}$$
$$\times\,\mathrm{Ai}\,\{\kappa^{\frac{2}{3}}(1+qy-\alpha^2)\,\mathrm{e}^{\pi\mathrm{i}}\}\,\mathrm{e}^{-\mathrm{i}\alpha kN_1x}\,\mathrm{d}\alpha \quad (72)$$

from (41). Ignoring the order term for the moment and putting $\alpha^2 = 1+qy+\beta$ we obtain

$$\int_{-qy}^{(1+qy)\epsilon} \frac{\exp\left[-\mathrm{i}\kappa\{\frac{2}{3}(qy_0-qy-\beta)^{\frac{3}{2}}+qx(1+qy+\beta)^{\frac{1}{2}}\}\right]}{4\pi^{\frac{1}{2}}\kappa^{\frac{1}{6}}(qy_0-qy-\beta)^{\frac{1}{4}}\,\mathrm{e}^{\frac{1}{12}\pi\mathrm{i}}(1+qy+\beta)^{\frac{1}{2}}}\,\mathrm{Ai}\,(\kappa^{\frac{2}{3}}\beta)\,\mathrm{d}\beta$$
$$= \frac{\kappa^{\frac{1}{6}}}{8\pi^{\frac{3}{2}}}\int_{-qy}^{(1+qy)\epsilon}\int_{-\infty}^{\infty} \frac{\exp\left[-\mathrm{i}\kappa\{\frac{2}{3}(qy_0-qy-\beta)^{\frac{3}{2}}+qx(1+qy+\beta)^{\frac{1}{2}}+\frac{1}{3}u^3+\beta u\}\right]}{(qy_0-qy-\beta)^{\frac{1}{4}}\,(1+qy+\beta)^{\frac{1}{2}}\,\mathrm{e}^{\frac{1}{12}\pi\mathrm{i}}}\,\mathrm{d}u\,\mathrm{d}\beta,$$

from (40). The asymptotic behaviour of such double integrals has been discussed (Jones & Kline 1958) and it has been shown that the behaviour is dictated by the critical points in the area of integration. The critical points consist of the points where f, given by

$$f \equiv -\tfrac{2}{3}(qy_0-qy-\beta)^{\frac{3}{2}}-qx(1+qy+\beta)^{\frac{1}{2}}-\tfrac{1}{3}u^3-\beta u,$$

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is stationary and the boundary points where a curve $f =$ constant is tangent to the boundary. The stationary points of f are given by

$$(qy_0-qy-\beta)^{\frac{1}{2}}-\tfrac{1}{2}qx(1+qy+\beta)^{-\frac{1}{2}}-u=0, \tag{73}$$

$$u^2+\beta=0. \tag{74}$$

Equation (74) shows that $\beta \leqslant 0$. If we put $u=(-\beta)^{\frac{1}{2}}$ in (73) we recover (46) with the upper sign and $\alpha^2 = 1+qy+\beta$; on the other hand $u=-(-\beta)^{\frac{1}{2}}$ gives (46) with the lower sign. The stationary points therefore are precisely the same as those given by (46) and, when they are present, supply the field of geometrical optics unless they are near the boundary of the area of integration. According to the theory of Jones & Kline the next term in the asymptotic expansion is lower in order by the factor $1/\kappa$. Since such a factor is negligible in comparison with the order term in (72) there is no point in taking more than the first term of the asymptotic expansion. If the stationary point is near the boundary this theory does not hold and further consideration is necessary; this can happen only near the boundary $\beta=-qy$ $(\alpha=1)$ since $\beta \leqslant 0$ at a stationary point.

Turning now to the critical boundary points we write the integral as

$$\int_{-qy}^{(1+qy)\epsilon}\int_{-\infty}^{\infty} g(\beta,u)\,\mathrm{e}^{\mathrm{i}\kappa f}\,\mathrm{d}\beta\,\mathrm{d}u.$$

Then the contribution of a critical boundary point (β_0, u_0) (assumed to be non-stationary) is

$$\frac{g(\beta_0,u_0)\,\pi^{\frac{1}{2}}\exp\{\frac{3}{4}\pi\mathrm{i}+\mathrm{i}\kappa f(\beta_0,u_0)\}}{\kappa^{\frac{3}{2}}\left[\left(\dfrac{1}{2}\dfrac{\partial^2 f}{\partial u^2}\right)^{\frac{1}{2}}\dfrac{\partial f}{\partial \beta}\right]_{\beta=\beta_0,\,u=u_0}}+O\left(\frac{1}{\kappa^{\frac{5}{2}}}\right)$$

with the understanding that $\partial^2 f/\partial u^2 = |\partial^2 f/\partial u^2|\,\mathrm{e}^{\pi\mathrm{i}}$ when it is negative. Now the curve $f=$ constant is tangent to the boundary $\beta=$ constant only if (74) is satisfied. Hence the critical boundary points are $(\beta_0, (-\beta_0)^{\frac{1}{2}})$ and $(\beta_0, -(-\beta_0)^{\frac{1}{2}})$ where $\beta_0=-qy$. Their contribution to the integral is

$$\frac{\exp\{\frac{2}{3}\pi\mathrm{i}-\mathrm{i}\kappa qx-\frac{2}{3}\mathrm{i}\kappa(qy_0)^{\frac{3}{2}}\}}{8\pi\kappa^{\frac{4}{3}}(qy_0)^{\frac{1}{4}}(qy)^{\frac{1}{4}}}\left\{\frac{\exp\{-\frac{2}{3}\mathrm{i}\kappa(qy)^{\frac{3}{2}}\}}{(qy_0)^{\frac{1}{2}}+(qy)^{\frac{1}{2}}-\frac{1}{2}qx}-\frac{\mathrm{i}\exp\{\frac{2}{3}\mathrm{i}\kappa(qy)^{\frac{3}{2}}\}}{(qy_0)^{\frac{1}{2}}-(qy)^{\frac{1}{2}}-\frac{1}{2}qx}\right\}. \tag{75}$$

Consequently the value of (72) is the geometrical optics field provided by the interval plus (75) all multiplied by a factor which is $1+O(1/\kappa^{\frac{2}{3}}q(y_0-y))$ or $1+O(1/\kappa^{\eta})$ at worst. Unless y is close to y_0 it can be taken as $1+O(1/\kappa^{\frac{2}{3}})$ effectively and it will be written in this form from now on. If there are no stationary points the value is (75) only. The case when a stationary point is near $\alpha=1$ will be dealt with later.

The interval $(-(1+qy)^{\frac{1}{2}}(1+\epsilon)^{\frac{1}{2}}, -1)$ gives similar results with the sign of x reversed.

In the interval $\alpha^2 \leqslant 1$ we use (47) in (70). The modified first term is

$$\int_{-1}^{1}\mathrm{Ai}\{\kappa^{\frac{2}{3}}(1+qy_0-\alpha^2)\,\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}\}\,\mathrm{Ai}\{\kappa^{\frac{2}{3}}(1+qy-\alpha^2)\,\mathrm{e}^{-\frac{1}{3}\pi\mathrm{i}}\}\exp(-\tfrac{1}{3}\pi\mathrm{i}-\mathrm{i}\alpha kN_1x)\,\mathrm{d}\alpha$$

$$=\int_{-1}^{1}\frac{\exp\{-\frac{2}{3}\mathrm{i}\kappa(1+qy_0-\alpha^2)^{\frac{3}{2}}+\frac{2}{3}\mathrm{i}\kappa(1+qy-\alpha^2)^{\frac{3}{2}}-\mathrm{i}\alpha kN_1x-\frac{1}{3}\pi\mathrm{i}\}}{4\pi\kappa^{\frac{1}{3}}(1+qy_0-\alpha^2)^{\frac{1}{4}}(1+qy-\alpha^2)^{\frac{1}{4}}}\left\{1+O\left(\frac{1}{\kappa^{\frac{2}{3}}(1+qy-\alpha^2)}\right)\right\}\mathrm{d}\alpha. \tag{76}$$

Theorems concerning the asymptotic behaviour of integrals of this type have been given by Erdélyi (1955). An interior stationary point provides the geometrical optics of incident rays going down to strike the boundary plus a correction which is $O(1/\kappa)$ smaller. If the stationary point is near the end of the interval the theory does not apply but since this does not occur near the shadow boundary we shall not consider the matter further. The contribution of the end-points is

$$\frac{e^{\frac{1}{6}\pi i}\exp\left[i\kappa\{-\frac{2}{3}(qy_0)^{\frac{3}{2}}+\frac{2}{3}(qy)^{\frac{3}{2}}+qx\}\right]}{4\pi\kappa^{\frac{4}{3}}(qy_0)^{\frac{1}{4}}(qy)^{\frac{1}{4}}\{-2(qy_0)^{\frac{1}{2}}+2(qy)^{\frac{1}{2}}-qx\}}-\frac{\exp\left[\frac{1}{6}\pi i+i\kappa\{-\frac{2}{3}(qy_0)^{\frac{3}{2}}+\frac{2}{3}(qy)^{\frac{3}{2}}-qx\}\right]}{4\pi\kappa^{\frac{4}{3}}(qy_0)^{\frac{1}{4}}(qy)^{\frac{1}{4}}\{2(qy_0)^{\frac{1}{2}}-2(qy)^{\frac{1}{2}}-qx\}}.$$

Combining this result with (75) we see that the contribution of the first term of (70) (with the change described just after (44)) to ψ consists of

$$\left[\gamma_i\,e^{-ikL_i}-\frac{N_1^{\frac{1}{2}}\,e^{\frac{3}{4}\pi i}}{(2\pi\kappa)^{\frac{1}{2}}\,q(yy_0)^{\frac{1}{4}}}\left\{\frac{e^{-ikL}}{L_b}+\frac{e^{-ik\bar{L}}}{\bar{L}_b}\right\}\right][1+O(\kappa^{-\frac{2}{3}})], \tag{77}$$

where $\bar{L}$ and $\bar{L}_b$ are the same as L and L_b respectively with $-x$ in place of x. The geometrical optics field $\gamma_i\,e^{-ikL_i}$ is of course absent in the shadow.

There is one exceptional case when (77) is not valid and that is when, as mentioned above, a solution of (73) and (74) is such that $\beta \approx -qy$. The root $u = (-\beta)^{\frac{1}{2}}$ provides a stationary point which is also one for (76) but just outside the interval of integration. When these two integrals are taken together the contribution of an ordinary stationary point, i.e. geometrical optics is obtained. The root $u = -(-\beta)^{\frac{1}{2}}$, however, has no connexion with (76) and must be examined separately. It arises only when $L_b \approx 0$, i.e. near the shadow boundary. In that case split the interval of integration in (72) into

$$(1, (1+\epsilon)^{\frac{1}{2}}) \quad \text{and} \quad ((1+\epsilon)^{\frac{1}{2}}, (1+qy)^{\frac{1}{2}}(1+\epsilon)^{\frac{1}{2}}).$$

Deal with the second interval in the same way as (72) was dealt with. In place of the first end-point term in (75) (which in fact becomes infinite as $L_b \to 0$) we obtain

$$\frac{\exp\{\frac{2}{3}\pi i-i\kappa qx(1+\epsilon)^{\frac{1}{2}}-\frac{2}{3}i\kappa(qy_0-\epsilon)^{\frac{3}{2}}-\frac{2}{3}i\kappa(qy-\epsilon)^{\frac{3}{2}}\}}{8\pi\kappa^{\frac{4}{3}}(qy_0-\epsilon)^{\frac{1}{4}}(qy-\epsilon)^{\frac{1}{4}}\{(qy_0-\epsilon)^{\frac{1}{2}}+(qy-\epsilon)^{\frac{1}{2}}-\frac{1}{2}qx(1+\epsilon)^{-\frac{1}{2}}\}}. \tag{78}$$

In the remaining interval the relation (47) can be used and the second Airy function discarded since it gives rise to the root $u = (-\beta)^{\frac{1}{2}}$ mentioned above. The other Airy function can be replaced by the first term of its asymptotic expansion so that

$$\int_1^{(1+\epsilon)^{\frac{1}{2}}}\frac{\exp\left[-\frac{2}{3}i\kappa\{(1+qy_0-\alpha^2)^{\frac{3}{2}}+(1+qy-\alpha^2)^{\frac{3}{2}}+\frac{3}{2}q\alpha x\}+\frac{1}{3}\pi i\right]}{4\pi\kappa^{\frac{1}{3}}(1+qy_0-\alpha^2)^{\frac{1}{4}}(1+qy-\alpha^2)^{\frac{1}{4}}\,e^{\frac{1}{6}\pi i}}\left\{1+O\left(\frac{1}{\kappa^{\frac{2}{3}}(1+qy-\alpha^2)}\right)\right\}d\alpha$$

is obtained. The substitution $\alpha^2 = 1+\beta$ gives, neglecting the order term for the moment,

$$\int_0^{\epsilon}\frac{\exp\left[-\frac{2}{3}i\kappa\{(qy_0-\beta)^{\frac{3}{2}}+(qy-\beta)^{\frac{3}{2}}+\frac{3}{2}qx(1+\beta)^{\frac{1}{2}}\}+\frac{1}{6}\pi i\right]}{8\pi\kappa^{\frac{1}{3}}(qy_0-\beta)^{\frac{1}{4}}(qy-\beta)^{\frac{1}{4}}(1+\beta)^{\frac{1}{2}}}\,d\beta$$

$$=\int_0^{\epsilon}\frac{\exp\{-ikL-\frac{1}{2}ikL_b\beta-\frac{1}{4}ikL_1\beta^2+\frac{1}{6}\pi i\}}{8\pi\kappa^{\frac{1}{3}}(q^2yy_0)^{\frac{1}{4}}}\{1+O(\ldots)\}\,d\beta, \tag{79}$$

where
$$L_1 = (N_1/q)\{(qy)^{-\frac{1}{2}}+(qy_0)^{-\frac{1}{2}}-\frac{1}{2}qx\}$$

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and the dots in the order term indicate terms such as β, β/qy, $\kappa\beta^3$. Since, for $L_1 > 0$

$$\int_0^\epsilon \exp\left(-\tfrac{1}{2}\mathrm{i}kL_b\beta - \tfrac{1}{4}\mathrm{i}kL_1\beta^2\right)\mathrm{d}\beta = \frac{2\exp\left(\frac{1}{4}\mathrm{i}kL_b^2/L_1\right)}{(kL_1)^{\frac{1}{2}}}\int_{\frac{1}{2}L_b(k/L_1)^{\frac{1}{2}}}^{\frac{1}{2}(\epsilon L_1+L_b)(k/L_1)^{\frac{1}{2}}} \mathrm{e}^{-\mathrm{i}\beta^2}\,\mathrm{d}\beta$$

and, as $a\to\infty$,
$$\int_a^\infty \mathrm{e}^{-\mathrm{i}\beta^2}\,\mathrm{d}\beta = \frac{\mathrm{e}^{-\mathrm{i}a^2}}{2\mathrm{i}a} + O\left(\frac{1}{a^3}\right), \tag{80}$$
the value of (79) is

$$\frac{\exp\{-\mathrm{i}k(L-\frac{1}{4}L_b^2/L_1)+\frac{1}{6}\pi\mathrm{i}\}}{4\pi\kappa^{\frac{1}{3}}(q^2yy_0)^{\frac{1}{4}}(kL_1)^{\frac{1}{2}}}\left\{\int_{\frac{1}{2}L_b(k/L_1)^{\frac{1}{2}}}^\infty \mathrm{e}^{-\mathrm{i}\beta^2}\mathrm{d}\beta + \frac{\mathrm{i}\exp\{-\frac{1}{4}\mathrm{i}k(L_1\epsilon+L_b)^2/L_1\}}{(k/L_1)^{\frac{1}{2}}(L_1\epsilon+L_b)}\right\}\{1+O(\kappa^{-\frac{2}{3}})\}$$

provided that $L_1\epsilon+L_b>0$. Combining this with (78) the total contribution to ψ in this case from the interval $\alpha \geqslant 1$ is

$$\frac{\exp\{\frac{1}{4}\pi\mathrm{i}-\mathrm{i}k(L-\frac{1}{4}L_b^2/L_1)\}}{(2\pi L_1)^{\frac{1}{2}}(q^2yy_0)^{\frac{1}{4}}}\int_{\frac{1}{2}L_b(k/L_1)^{\frac{1}{2}}}^\infty \mathrm{e}^{-\mathrm{i}\beta^2}\,\mathrm{d}\beta\{1+O(\kappa^{-\frac{2}{3}})\}. \tag{81}$$

The same formula is true for $L_1<0$ and $L_1\epsilon+L_b<0$ provided that L_1 is then understood to mean $|L_1|\,\mathrm{e}^{\pi\mathrm{i}}$. Since $L_b\approx 0$, $L_1>0$ implies that the point of observation is below the point where the shadow boundary and caustic touch whereas $L_1<0$ means that the point of observation is above the point of tangency.

9. The reflected wave term

We turn now to the second integral of (70). We have

$$\int_1^{(1+\epsilon)^{\frac{1}{2}}} R_1(\alpha^2)\,\mathrm{Ai}\{\kappa^{\frac{2}{3}}(1+qy_0-\alpha^2)\,\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}\}\,\mathrm{Ai}\{\kappa^{\frac{2}{3}}(1+qy-\alpha^2)\,\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}\}\,\mathrm{e}^{-\mathrm{i}\alpha kN_1x}\,\mathrm{d}\alpha$$

$$=\int_1^{(1+\epsilon)^{\frac{1}{2}}}\frac{R_1(\alpha^2)\exp[-\mathrm{i}\kappa\{\frac{2}{3}(1+qy_0-\alpha^2)^{\frac{3}{2}}+\frac{2}{3}(1+qy-\alpha^2)^{\frac{3}{2}}+\alpha qx\}]}{4\pi\kappa^{\frac{1}{3}}\,\mathrm{e}^{\frac{1}{6}\pi\mathrm{i}}(1+qy_0-\alpha^2)^{\frac{1}{4}}(1+qy-\alpha^2)^{\frac{1}{4}}}\left\{1+O\left(\frac{1}{\kappa^{\frac{2}{3}}(1+qy-\alpha^2)}\right)\right\}\mathrm{d}\alpha.$$

Ignoring the order term and putting $\alpha^2 = 1+\beta$ we obtain

$$\int_0^\epsilon \frac{R_1(1+\beta)\exp[-\mathrm{i}\kappa\{\frac{2}{3}(qy_0-\beta)^{\frac{3}{2}}+\frac{2}{3}(qy-\beta)^{\frac{3}{2}}+(1+\beta)^{\frac{1}{2}}qx\}]}{8\pi\kappa^{\frac{1}{3}}\,\mathrm{e}^{\frac{1}{6}\pi\mathrm{i}}(qy_0-\beta)^{\frac{1}{4}}(qy-\beta)^{\frac{1}{4}}(1+\beta)^{\frac{1}{2}}}\,\mathrm{d}\beta$$
$$=\int_0^\epsilon \frac{R_1(1+\beta)\exp(-\mathrm{i}kL-\frac{1}{2}\mathrm{i}kL_b\beta)}{8\pi\kappa^{\frac{1}{3}}\,\mathrm{e}^{\frac{1}{6}\pi\mathrm{i}}(qy_0)^{\frac{1}{4}}(qy)^{\frac{1}{4}}}\{1-\tfrac{1}{4}\mathrm{i}kL_1\beta^2+O(\ldots)\}\,\mathrm{d}\beta,$$

where the dots in the order bracket indicate terms such as β, β/qy, $\kappa\beta^3$. Since

$$\int_0^\epsilon R_1(1+\beta)(1-\tfrac{1}{4}\mathrm{i}kL_1\beta^2)\,\mathrm{e}^{-\frac{1}{2}\mathrm{i}kL_b\beta}\,\mathrm{d}\beta = \int_0^\infty R_1(1+\beta)(1-\tfrac{1}{4}\mathrm{i}kL_1\beta^2)\,\mathrm{e}^{-\frac{1}{2}\mathrm{i}kL_b\beta}\,\mathrm{d}\beta + O\{\exp(-\kappa^{\frac{2}{3}}\eta)\}$$

our final result is

$$-\frac{\mathrm{e}^{-\frac{1}{6}\pi\mathrm{i}-\mathrm{i}kL}}{8\pi\kappa(q^2yy_0)^{\frac{1}{4}}}\int_0^\infty \frac{\mathrm{e}^{\pi\mathrm{i}}\,\mathrm{Ai}'(\mu)+Z\,\mathrm{Ai}(\mu)}{\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}\,\mathrm{Ai}'(\mu\,\mathrm{e}^{-\frac{2}{3}\pi\mathrm{i}})+Z\,\mathrm{Ai}(\mu\,\mathrm{e}^{-\frac{2}{3}\pi\mathrm{i}})}$$
$$\times\left(1-\frac{1}{4}\frac{\mathrm{i}N_1L_1}{q\kappa^{\frac{1}{3}}}\mu^2\right)\exp(-\tfrac{1}{2}\mathrm{i}kL_b\mu/\kappa^{\frac{2}{3}})\,\mathrm{d}\mu\{1+O(\kappa^{-\frac{2}{3}})\}. \tag{82}$$

There is also exactly the same contribution from the interval $(-(1+\epsilon)^{\frac{1}{2}},-1)$ except that the sign of x is reversed.

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Now consider the integral in $\alpha^2 \leqslant 1$ with the modification due to (47). It is

$$\int_{-1}^{1} R_2(\alpha^2)\,\mathrm{Ai}\,\{\kappa^{\frac{2}{3}}(1+qy_0-\alpha^2)\,\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}\}\,\mathrm{Ai}\,\{\kappa^{\frac{2}{3}}(1+qy-\alpha^2)\,\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}\}\,\mathrm{e}^{-\mathrm{i}\alpha kN_1x}\,\mathrm{d}\alpha,$$

where $$R_2(\alpha^2) = -\frac{\mathrm{e}^{-\frac{2}{3}\pi\mathrm{i}}\,\mathrm{Ai}'\,\{\kappa^{\frac{2}{3}}(1-\alpha^2)\,\mathrm{e}^{-\frac{1}{3}\pi\mathrm{i}}\}+Z\,\mathrm{e}^{-\frac{1}{3}\pi\mathrm{i}}\,\mathrm{Ai}\,\{\kappa^{\frac{2}{3}}(1-\alpha^2)\,\mathrm{e}^{-\frac{1}{3}\pi\mathrm{i}}\}}{\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}\,\mathrm{Ai}'\,\{\kappa^{\frac{2}{3}}(1-\alpha^2)\,\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}\}+Z\,\mathrm{Ai}\,\{\kappa^{\frac{2}{3}}(1-\alpha^2)\,\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}\}}.$$

In the interval $\{-(1-\epsilon)^{\frac{1}{2}}, (1-\epsilon)^{\frac{1}{2}}\}$ asymptotic formulae may be employed to give

$$\int_{-(1-\epsilon)^{\frac{1}{2}}}^{(1-\epsilon)^{\frac{1}{2}}} \frac{\mathrm{e}^{-\frac{1}{2}\pi\mathrm{i}}\,\kappa^{\frac{1}{3}}(1-\alpha^2)^{\frac{1}{2}}-Z}{Z-\mathrm{e}^{\frac{1}{2}\pi\mathrm{i}}\,\kappa^{\frac{1}{3}}(1-\alpha^2)^{\frac{1}{2}}}$$
$$\times\frac{\exp\{-\frac{1}{3}\pi\mathrm{i}-\frac{2}{3}\mathrm{i}\kappa(1+qy_0-\alpha^2)^{\frac{3}{2}}-\frac{2}{3}\mathrm{i}\kappa(1+qy-\alpha^2)^{\frac{3}{2}}-\mathrm{i}\alpha kN_1x+\frac{4}{3}\mathrm{i}\kappa(1-\alpha^2)^{\frac{3}{2}}\}}{4\pi\kappa^{\frac{1}{3}}(1+qy_0-\alpha^2)^{\frac{1}{4}}\,(1+qy-\alpha^2)^{\frac{1}{4}}}$$
$$\times\left\{1+O\left(\frac{1}{\kappa^{\frac{2}{3}}(1-\alpha^2)}\right)\right\}\mathrm{d}\alpha.$$

A stationary point in this interval provides the waves reflected from the boundary by the laws of geometrical optics. The contribution of the end-points is

$$-\frac{Z+\mathrm{i}\kappa^{\frac{1}{3}}\epsilon^{\frac{1}{2}}}{Z-\mathrm{i}\kappa^{\frac{1}{3}}\epsilon^{\frac{1}{2}}}\,\frac{\exp\,(-\frac{1}{3}\pi\mathrm{i}+\frac{4}{3}\mathrm{i}\kappa\epsilon^{\frac{3}{2}})\,N_1}{4\pi\mathrm{i}\kappa^{\frac{4}{3}}q(q^2y_0y)^{\frac{1}{4}}}$$
$$\times\left\{\frac{(1-\frac{1}{4}\mathrm{i}kL_1\epsilon^2)\exp\,(-\mathrm{i}kL+\frac{1}{2}\mathrm{i}kL_b\epsilon)}{-L_b-4\epsilon^{\frac{1}{2}}N_1/q}+\frac{(1-\frac{1}{4}\mathrm{i}k\overline{L}_1\epsilon^2)\exp\,(-\mathrm{i}k\overline{L}+\frac{1}{2}\mathrm{i}k\overline{L}_b\epsilon)}{(\overline{L}_1+\frac{1}{2}\overline{L}_b)\,\epsilon-\overline{L}_b+4\epsilon^{\frac{1}{2}}N_1/q}\right\}, \tag{83}$$

where $\overline{L}$, $\overline{L}_1$, $\overline{L}_b$ are the same as L, L_1, L_b, respectively, with x instead of $-x$.

With regard to the interval $((1-\epsilon)^{\frac{1}{2}}, 1)$ put $\alpha^2 = 1-\beta$ to obtain

$$\int_0^{\epsilon} R_2(1-\beta)\,\frac{\exp\,[-\frac{1}{6}\pi\mathrm{i}-\mathrm{i}\kappa\{\frac{2}{3}(qy_0+\beta)^{\frac{3}{2}}+\frac{2}{3}(qy+\beta)^{\frac{3}{2}}+(1-\beta)\,qx\}]}{8\pi\kappa^{\frac{1}{3}}(qy_0+\beta)^{\frac{1}{4}}\,(qy+\beta)^{\frac{1}{4}}\,(1-\beta)^{\frac{1}{2}}}\left\{1+O\left(\frac{1}{\kappa^{\frac{2}{3}}(qy+\beta)}\right)\right\}\mathrm{d}\beta.$$

Ignoring the order term for the present the integral can be written

$$\int_0^{\epsilon} R_2(1-\beta)\,\frac{(1-\frac{1}{4}\mathrm{i}kL_1\beta^2)\exp\{-\frac{1}{6}\pi\mathrm{i}-\mathrm{i}k(L-\frac{1}{2}L_b\beta)\}}{8\pi\kappa^{\frac{1}{3}}(q^2yy_0)^{\frac{1}{4}}}\{1+O(\ldots)\}\,\mathrm{d}\beta,$$

where the dots in the order term indicate term such as β, β/qy. Now

$$\int_0^{\epsilon} R_2(1-\beta)\,\mathrm{e}^{\mathrm{i}k(\frac{1}{2}L_b\beta)}\,\mathrm{d}\beta = \int_0^{\infty} R_2(1-\beta)\,\mathrm{e}^{\mathrm{i}k(\frac{1}{2}L_b\beta)}\,\mathrm{d}\beta-\frac{Z+\mathrm{i}\kappa^{\frac{1}{3}}\epsilon^{\frac{1}{2}}}{Z-\mathrm{i}\kappa^{\frac{1}{3}}\epsilon^{\frac{1}{2}}}\,\frac{N_1\exp\,(-\frac{1}{6}\pi\mathrm{i}+\frac{4}{3}\mathrm{i}\kappa\epsilon^{\frac{3}{2}}+\frac{1}{2}\mathrm{i}kL_b\epsilon)}{\frac{1}{2}\mathrm{i}\kappa q(4N_1\epsilon^{\frac{1}{2}}/q+L_b)} \tag{84}$$

provided that $qL_b+4N_1\epsilon^{\frac{1}{2}}>0$. If $qL_b+4N_1\epsilon^{\frac{1}{2}}<0$ there is a contribution from a stationary point in (ϵ,∞) to be subtracted. Also

$$\int_0^{\epsilon} R_2(1-\beta)\,\beta^2\,\mathrm{e}^{\frac{1}{2}\mathrm{i}kL_b\beta}\,\mathrm{d}\beta = -\frac{\partial^2}{\partial(\frac{1}{2}kL_b)^2}\int_0^{\epsilon} R_2(1-\beta)\,\mathrm{e}^{\frac{1}{2}\mathrm{i}kL_b\beta}\,\mathrm{d}\beta$$
$$= -\frac{\partial^2}{\partial(\frac{1}{2}kL_b)^2}\int_0^{\infty} R_2(1-\beta)\,\mathrm{e}^{\frac{1}{2}\mathrm{i}kL_b\beta}\,\mathrm{d}\beta-\frac{Z+\mathrm{i}\kappa^{\frac{1}{3}}\epsilon^{\frac{1}{2}}}{Z-\mathrm{i}\kappa^{\frac{1}{3}}\epsilon^{\frac{1}{2}}}\,\frac{\epsilon^2N_1\exp\,(-\frac{1}{6}\pi\mathrm{i}+\frac{4}{3}\mathrm{i}\kappa^{\frac{3}{2}}+\frac{1}{2}\mathrm{i}kL_b\epsilon)}{\frac{1}{2}\mathrm{i}\kappa q(4N_1\epsilon^{\frac{1}{2}}/q+L_b)}, \tag{85}$$

together with a contribution from the stationary point if $qL_b+4N_1\epsilon^{\frac{1}{2}}<0$.

The first term in the brackets of (83) cancels the sum of the last terms of (84) and (85). Hence the net result of the integration over the interval $(-1, 1)$ is to provide a contribution to ψ of

$$\gamma_r \mathrm{e}^{-\mathrm{i}kL_r} + \frac{BkN_1}{16\pi^2\kappa^{\frac{1}{3}}} \frac{\mathrm{e}^{-\frac{1}{6}\pi\mathrm{i}}}{(q^2yy_0)^{\frac{1}{4}}} \left[\mathrm{e}^{-\mathrm{i}kL} \left\{ 1 + \tfrac{1}{4}\mathrm{i}kL_1 \frac{\partial^2}{\partial(\frac{1}{2}kL_b)^2} \right\} \int_0^\infty R_2(1-\beta)\, \mathrm{e}^{\frac{1}{2}\mathrm{i}kL_b\beta}\, \mathrm{d}\beta \right.$$
$$\left. + \mathrm{e}^{-\mathrm{i}k\bar{L}} \left\{ 1 + \tfrac{1}{4}\mathrm{i}k\bar{L}_1 \frac{\partial^2}{\partial(\frac{1}{2}kL_b)^2} \right\} \int_0^\infty R_2(1-\beta)\, \mathrm{e}^{\frac{1}{2}\mathrm{i}k\bar{L}_b\beta}\, \mathrm{d}\beta \right] \{1 + O(\kappa^{-\frac{2}{3}})\} \qquad (86)$$

with the understanding that the contributions of the stationary points in the integrals are to be deleted when $qL_b + 4N_1\epsilon^{\frac{1}{2}} < 0$. The term $\gamma_r \mathrm{e}^{-\mathrm{i}kL_r}$ is the wave reflected from the boundary according to the laws of geometrical optics and is non-zero for $qL_b + 4N_1\epsilon^{\frac{1}{2}} < 0$.

Combining (81), (86) and the term in (77) left by the process leading to (81) we obtain for the field near the shadow boundary $qL_b + 4N_1\epsilon^{\frac{1}{2}} > 0$ and below the caustic

$$\psi = \frac{\exp\{-\frac{1}{4}\pi\mathrm{i} - \mathrm{i}k(L - \frac{1}{4}L_b^2/L_1)\}}{(2\pi L_1)^{\frac{1}{2}}(q^2yy_0)^{\frac{1}{4}}} \int_{\frac{1}{2}L_b(k/L_1)^{\frac{1}{2}}}^\infty \mathrm{e}^{-\mathrm{i}\beta^2}\, \mathrm{d}\beta - \frac{N_1^{\frac{1}{2}} \mathrm{e}^{\frac{3}{4}\pi\mathrm{i} - \mathrm{i}k\bar{L}}}{2^{\frac{1}{2}}\pi^{\frac{1}{2}}\kappa^{\frac{1}{2}}q^{\frac{1}{2}}(q^2yy_0)^{\frac{1}{4}}\bar{L}_b}$$
$$- \frac{q^{\frac{1}{6}}\mathrm{e}^{-\mathrm{i}kL}}{2^{\frac{1}{2}}k^{\frac{1}{6}}N_1^{\frac{2}{3}}(yy_0)^{\frac{1}{4}}} \left[G\{\tfrac{1}{2}kL_b\kappa^{-\frac{2}{3}}\} + \frac{1}{4}\frac{\mathrm{i}kL_1}{\kappa^{\frac{4}{3}}} G''\{\tfrac{1}{2}kL_b\kappa^{-\frac{2}{3}}\} \right]$$
$$- \frac{q^{\frac{1}{6}}\mathrm{e}^{-\mathrm{i}k\bar{L}}}{2^{\frac{1}{2}}k^{\frac{1}{6}}N_1^{\frac{2}{3}}(yy_0)^{\frac{1}{4}}} \left[G\{\tfrac{1}{2}k\bar{L}_b\kappa^{-\frac{2}{3}}\} + \frac{1}{4}\frac{\mathrm{i}k\bar{L}_1}{\kappa^{\frac{4}{3}}} G''\{\tfrac{1}{2}kL_b\kappa^{-\frac{2}{3}}\} \right], \qquad (87)$$

where

$$G(\tau) = \frac{\mathrm{e}^{-\frac{1}{12}\pi\mathrm{i}}}{2\pi^{\frac{1}{2}}} \int_0^\infty \frac{\mathrm{e}^{\pi\mathrm{i}}\,\mathrm{Ai}'(\mu) + Z\,\mathrm{Ai}(\mu)}{\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}\,\mathrm{Ai}'(\mu\,\mathrm{e}^{-\frac{2}{3}\pi\mathrm{i}}) + Z\,\mathrm{Ai}(\mu\,\mathrm{e}^{-\frac{2}{3}\pi\mathrm{i}})} \mathrm{e}^{-\mathrm{i}\mu\tau}\, \mathrm{d}\mu$$
$$+ \frac{\mathrm{e}^{-\frac{5}{12}\pi\mathrm{i}}}{2\pi^{\frac{1}{2}}} \int_0^\infty \frac{\mathrm{e}^{-\frac{1}{3}\pi\mathrm{i}}\,\mathrm{Ai}'(\mu\,\mathrm{e}^{-\frac{1}{3}\pi\mathrm{i}}) + Z\,\mathrm{Ai}(\mu\,\mathrm{e}^{-\frac{1}{3}\pi\mathrm{i}})}{\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}\,\mathrm{Ai}'(\mu\,\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}) + Z\,\mathrm{Ai}(\mu\,\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}})} \mathrm{e}^{\mathrm{i}\mu\tau}\, \mathrm{d}\mu \qquad (88)$$

and $G''(\tau) = \mathrm{d}^2G/\mathrm{d}\tau^2$. The whole of (87) is multiplied by $1 + O(\kappa^{-\frac{2}{3}})$.

In the shadow the first two terms of (87) are replaced by (77) with ψ_i absent. In the illuminated region the first two terms of (87) are replaced by (77), ψ_r is added and the convention on the stationary points in G is adopted.

Thus the field has been obtained everywhere, except in the caustic region, with a relative error of $O(\kappa^{-\frac{2}{3}})$. The formula (87) will now be simplified.

10. A simplified formula for the field

In deriving a simplified version of the field various results for the function G are used; they will be found in appendix A and reference to equation (5), say, of that appendix will be denoted by (A 5).

First note that $\frac{1}{2}k\bar{L}_b\kappa^{-\frac{2}{3}} < -\kappa^{\frac{1}{3}}\epsilon^{\frac{1}{2}}$ since only the case $x > 0$ is being considered. Therefore the asymptotic formula (A 7) and (A 9) may be used, the first term of (A 7) being omitted in view of our convention on the stationary points. Thus

$$G(\tfrac{1}{2}k\bar{L}_b\kappa^{-\frac{2}{3}}) + \frac{1}{4}\frac{\mathrm{i}k\bar{L}_1}{\kappa^{\frac{4}{3}}} G''(\tfrac{1}{2}k\bar{L}_b\kappa^{-\frac{2}{3}}) = \frac{\kappa^{\frac{2}{3}}\mathrm{e}^{-\frac{1}{4}\pi\mathrm{i}}}{\pi^{\frac{1}{2}}k\bar{L}_b} + O(\kappa^{-\frac{2}{3}}). \qquad (89)$$

The substitution of this result in (87) provides a term which cancels the second term. Consequently (87) reduces to

$$\psi = \frac{\exp\{\frac{1}{4}\pi\mathrm{i} - \mathrm{i}k(L - \frac{1}{4}L_b^2/L_1)\}}{(2\pi L_1)^{\frac{1}{2}}\,(q^2yy_0)^{\frac{1}{4}}} \int_{\frac{1}{2}L_b(k/L_1)^{\frac{1}{2}}}^{\infty} \mathrm{e}^{-\mathrm{i}\beta^2}\,\mathrm{d}\beta - \frac{q^{\frac{1}{6}}\,\mathrm{e}^{-\mathrm{i}kL}}{2^{\frac{1}{2}}k^{\frac{1}{6}}N_1^{\frac{2}{3}}(yy_0)^{\frac{1}{4}}}\left[G(\tfrac{1}{2}kL_b\kappa^{-\frac{2}{3}}) + \frac{\mathrm{i}kL_1}{4\kappa^{\frac{4}{3}}}G''(\tfrac{1}{2}kL_b\kappa^{-\frac{2}{3}})\right]. \tag{90}$$

In the illuminated region the first term is replaced by the first two terms of (77) and $\gamma_r\,\mathrm{e}^{-\mathrm{i}kL_r}$ is added. In this region (89) may be used with $\overline{L}_b$ replaced by L_b. The result is

$$\psi = \gamma_i\,\mathrm{e}^{-\mathrm{i}kL_i} + \gamma_r\,\mathrm{e}^{-\mathrm{i}kL_r} + O(\kappa^{-\frac{2}{3}}). \tag{91}$$

In the shadow the first term of (90) is replaced by the second term of (77). The expressions (A 5) and (A 8) can be used with the result

$$\psi = \frac{q^{\frac{1}{6}}\exp(-\frac{1}{12}\pi\mathrm{i} - \mathrm{i}kL)}{2^{\frac{3}{2}}k^{\frac{1}{6}}\pi^{\frac{1}{2}}N_1^{\frac{2}{3}}(yy_0)^{\frac{1}{4}}} \sum_{s=1} \frac{Z^2(1 - \mathrm{i}kL_1\delta_s^2\,\mathrm{e}^{-\frac{2}{3}\pi\mathrm{i}}/4\kappa^{\frac{4}{3}})\exp(\frac{1}{2}\mathrm{i}\delta_s kL_b\kappa^{-\frac{2}{3}}\,\mathrm{e}^{-\frac{1}{3}\pi\mathrm{i}})}{(Z^2 - \delta_s\,\mathrm{e}^{\frac{2}{3}\pi\mathrm{i}})\,\{\mathrm{Ai}\,(\delta_s)\}^2} + O\left(\frac{1}{\kappa}\right). \tag{92}$$

It may be anticipated from the results of § 6 that the order term in fact contains an exponentially damped factor.

Now let us consider what happens to (90) as L_b increases in magnitude from zero. When it is sufficiently large and positive we can employ (80), (A 5) and (A 8). Equation (92) is at once recovered. On the other hand when L_b is sufficiently negative the formula

$$\int_{-a}^{\infty} \mathrm{e}^{-\mathrm{i}\beta^2}\,\mathrm{d}\beta = \pi^{\frac{1}{2}}\,\mathrm{e}^{-\frac{1}{4}\pi\mathrm{i}} - \frac{\mathrm{e}^{-\mathrm{i}a^2}}{2\mathrm{i}a} + O(a^{-3}) \tag{93}$$

and (A 7) can be used to give

$$\psi = \frac{\exp\{-\mathrm{i}k(L - \frac{1}{4}L_b^2/L_1)\}}{(2L_1)^{\frac{1}{2}}\,(q^2yy_0)^{\frac{1}{4}}} - \frac{Z - \frac{1}{4}\mathrm{i}kL_b\kappa^{-\frac{2}{3}}}{Z + \frac{1}{4}\mathrm{i}kL_b\kappa^{-\frac{2}{3}}}(-L_b)^{\frac{1}{2}}\frac{q^{\frac{1}{2}}\exp(-\mathrm{i}kL + \frac{1}{96}\mathrm{i}kL_b^3q^2/N_1^2)}{4N_1(yy_0)^{\frac{1}{4}}} + O(\kappa^{-\frac{2}{3}}). \tag{94}$$

In deriving this the first term of (A 7) was retained (contrary to our convention). The term G'' plays no part since (with our convention) its contribution would be contained in the order term.

Near the shadow boundary and below the caustic (32) gives, since $A \approx N_1$,

$$\gamma_i^2 = \frac{\frac{1}{2}q^{\frac{1}{2}}}{N_1(y^{\frac{1}{2}} + y_0^{\frac{1}{2}})\,\{1 - (q^2yy_0)^{\frac{1}{2}}\}} = \frac{1}{2L_1(q^2yy_0)^{\frac{1}{2}}} \tag{95}$$

since L_b is small. Also by expanding (30) and (31) in terms of $A - N_1$ about $A = N_1$ we find

$$L_i = L - \tfrac{1}{4}L_b^2/L_1. \tag{96}$$

Similarly, from (33), (35) and (36)

$$L_r = L - q^2L_b^3/96N_1^2, \tag{97}$$

$$\cos\chi = -qL_b/4N_1, \tag{98}$$

$$\gamma_r^2 = \frac{-R^2(\chi)\,q^2L_b}{16N_1^2(q^2yy_0)^{\frac{1}{2}}}. \tag{99}$$

On using (95) to (99), (94) can be written

$$\psi = \gamma_i \,\mathrm{e}^{-\mathrm{i}kL_i} + \gamma_r \,\mathrm{e}^{-\mathrm{i}kL_r} + O(\kappa^{-\frac{2}{3}}). \tag{100}$$

Although as L_b becomes increasingly negative, (94) does not coincide with (91), if converted to the form (100) before L_b becomes too negative it gives the correct field in the illuminated region. By this means (90) may be employed to give the correct field everywhere.

The formula takes a conveniently simple form if the approximations (95) and (96) are used. Then

$$\psi = \psi_i \frac{\mathrm{e}^{\frac{1}{4}\pi \mathrm{i}}}{\pi^{\frac{1}{2}}} \int_{\{k(L-L_i)\}^{\frac{1}{2}}}^{\infty} \mathrm{e}^{-\mathrm{i}\beta^2}\,\mathrm{d}\beta - \frac{q^{\frac{1}{6}}\,\mathrm{e}^{-\mathrm{i}kL}}{2^{\frac{1}{2}} k^{\frac{1}{6}} N_1^{\frac{2}{3}} (yy_0)^{\frac{1}{4}}} G(\tfrac{1}{2}kL_b\kappa^{-\frac{2}{3}}) + O(\kappa^{-\frac{2}{3}}). \tag{101}$$

In this formula ψ_i is the incident field $\gamma_i \,\mathrm{e}^{-\mathrm{i}kL_i}$ as given by the formulae (31) and (32) of geometrical optics. The square root $(L-L_i)^{\frac{1}{2}}$ is taken to be positive in the shadow and negative in the illuminated region. The function G is defined by (88) and the convention on the stationary point is dropped so that for large negative τ (A 7) in its entirety is used. Then (101) gives the field everywhere below the caustic provided that one starts at the shadow boundary ($L = L_i$, $L_b = 0$). As one moves away towards the illuminated region (101) soon takes the form (100) and from then on (100) is employed. As one moves away towards the shadow (101) soon takes the form

$$\psi = \left(\frac{k}{8\pi}\right)^{\frac{1}{2}} \frac{\exp(-\frac{1}{12}\pi \mathrm{i} - \mathrm{i}kL)}{\kappa^{\frac{2}{3}}(q^2 yy_0)^{\frac{1}{4}}} \sum_s \frac{Z^2 \exp(\frac{1}{2}\mathrm{i}kL_b\delta_s \,\mathrm{e}^{-\frac{1}{3}\pi \mathrm{i}}\kappa^{-\frac{2}{3}})}{(Z^2 - \delta_s \,\mathrm{e}^{\frac{2}{3}\pi \mathrm{i}})\{\mathrm{Ai}'(\delta_s)\}^2} + O(\kappa^{-\frac{2}{3}}) \tag{102}$$

$$= (\tfrac{1}{2}k)^{\frac{1}{2}} \frac{\mathrm{e}^{-\mathrm{i}kL}}{\kappa^{\frac{2}{3}}(q^2 yy_0)^{\frac{1}{4}}} \left\{\frac{\mathrm{e}^{-\frac{1}{4}\pi \mathrm{i}}}{\kappa^{-\frac{2}{3}} k \pi^{\frac{1}{2}} L_b} - G(\tfrac{1}{2}kL_b\kappa^{-\frac{2}{3}})\right\} \tag{103}$$

and from then on this form is used.

Note that (102) reproduces (60). Moreover, since $G(0)$ certainly exists, (A 5) shows that this field becomes infinite by the factor $1/L_b$ as $L_b \to 0$. Thus the Seckler & Keller theory of diffraction can be valid in the shadow only where $kL_b\kappa^{-\frac{2}{3}}$ is large.

11. Invariance under conformal mapping

Formula (101) gives the field everywhere below the caustic. In this section it will be rewritten so as to retain only those features of our model which are applicable to other media and other boundaries. In this way it is hoped to derive a formula for the field of wide applicability.

First, note that on the ray from the source which just touches the boundary

$$\gamma_i = (\tfrac{1}{2}q/N_1)^{\frac{1}{3}}/(qy_0)^{\frac{1}{4}}$$

at the boundary. Denote this by γ_A. Similarly, a source at the point of observation would produce an amplitude at the boundary on the ray of glancing incidence of $(\frac{1}{2}q/N_1)^{\frac{1}{3}}/(qy)^{\frac{1}{4}}$; denote this by γ_B.

Secondly, the radius of curvature ϱ of a ray is given by

$$1/\varrho = \mathbf{n}.\mathrm{grad}\ln N, \tag{104}$$

where $\mathbf{n}$ is the unit vector normal to the ray. For the particular medium under consideration the radius of curvature of a ray of glancing incidence is

$$\frac{1}{\varrho} = \frac{\partial}{\partial y}\ln N = \tfrac{1}{2}q$$

at the boundary.

The formula for ψ may now be written

$$\psi = \psi_i \frac{e^{\frac{1}{4}\pi i}}{\sqrt{\pi}} \int_{\{k(L-L_i)\}^{\frac{1}{2}}}^{\infty} e^{-i\beta^2} d\beta - (\tfrac{1}{2}\varrho N_1)^{\frac{1}{3}} \frac{\gamma_A \gamma_B \sqrt{2}\, e^{-ikL}}{k^{\frac{1}{6}}} G\left\{\tfrac{1}{2}k^{\frac{1}{3}} L_b \left(\frac{2}{N_1 \varrho}\right)^{\frac{2}{3}}\right\}. \tag{105}$$

This formula has the advantage that it carries no specific reference to the particular medium under consideration. It involves only optical path lengths, amplitudes, the refractive index and radius of curvature at the boundary. It is therefore possible that this formula is of wide applicability.

To examine this possibility make the conformal mapping $w = f(z)$ where $w = u + iv$ and $z = x + iy$. The equation (3) becomes

$$\frac{\partial^2 \psi}{\partial u^2} + \frac{\partial^2 \psi}{\partial v^2} + k^2 N^{(1)2} \psi = 0,$$

where $N^{(1)} = N|dz/dw|$. Thus the effect of the conformal mapping on (105) is to provide a field in a different inhomogeneous medium satisfying the transformed boundary condition on the image of the boundary $y = 0$. Now optical path length satisfies

$$\begin{aligned} L = \int N ds &= \int N |dz| \\ &= \int N |dz/dw|\, |dw| \\ &= \int N^{(1)} |dw| \\ &= L^{(1)}, \end{aligned}$$

say. Thus the curves L = constant become the curves $L^{(1)}$ = constant and, furthermore, $\text{grad}^{(1)2} L^{(1)} = N^{(1)2}$ where $\text{grad}^{(1)}$ is the gradient in the w-plane. As a consequence wavefronts are mapped into wavefronts and the rays, their orthogonal trajectories, are mapped into rays on account of the conformal property of the mapping. Hence optical path length is invariant under a conformal mapping. Also, if $\delta\sigma$ is the perpendicular distance between two rays, $N\delta\sigma$ transforms into $N^{(1)}\delta\sigma^{(1)}$ so that the amplitude of geometrical optics is also invariant under a conformal mapping.

Hence the effect of a conformal mapping on ψ_i, L, L_i, γ_A and γ_B in ψ is to replace them by $\psi_i^{(1)}$, $L^{(1)}$, $L_i^{(1)}$, $\gamma_A^{(1)}$ and $\gamma_B^{(1)}$, respectively, where these quantities are calculated directly in the new medium, the amplitudes being produced by a line source which produces a field $\exp(-ikN_0^{(1)} r^{(1)}/(N_0^{(1)} r^{(1)})^{\frac{1}{2}}$ at nearby points. (The affix (1) always indicates quantities in the w-plane.)

With regard to the radius of curvature of a ray

$$\begin{aligned} 1/\varrho &= \mathbf{n}.\text{grad} \ln N \\ &= \left|\frac{dw}{dz}\right| \mathbf{n}^{(1)}.\text{grad}^{(1)} \ln N \\ &= \left|\frac{dw}{dz}\right| \mathbf{n}^{(1)}.\left\{\text{grad}^{(1)} \ln N^{(1)} - \text{grad}^{(1)} \ln \left|\frac{dz}{dw}\right|\right\}. \end{aligned}$$

Now $|dz/dw|$ can be regarded as the refractive index in a medium obtained by conformal mapping from a homogeneous medium. In such a medium $y = 0$ would be a ray and so the boundary in the w-plane would be a ray so that $\mathbf{n}^{(1)}.\mathrm{grad}^{(1)} \ln |dz/dw|$ is the curvature the boundary $1/\varrho_1^{(1)}$. Hence

$$\frac{1}{N\varrho} = \frac{1}{N^{(1)}}\left\{\frac{1}{\varrho^{(1)}} - \frac{1}{\varrho_1^{(1)}}\right\}.$$

An immediate deduction is that $1/N\rho$ is an invariant of a conformal mapping where

$$\frac{1}{\rho} = \frac{1}{\varrho} - \frac{1}{\varrho_1}, \tag{106}$$

where ϱ and ϱ_1 are the radii of curvature of the ray of glancing incidence and boundary respectively, N being the refractive index at this point. The radii of curvature ϱ and ϱ_1 have the same sign when the centres of curvature are on the same side of the tangent at the point of glancing incidence, being positive when the centres of curvature are on the same side as the inhomogeneous medium.

It would not, however, be correct to replace $1/N_1\varrho$ in (105) by $1/N_A\rho_A$ (the suffix A indicating values at A) because this would allot a special position to A. On account of the reciprocity theorem that the point of observation and source can be interchanged without altering the value of the field A and B must appear symmetrically. Therefore $N_1\varrho$ is replaced by $(N_A\rho_A N_B\rho_B)^{\frac{1}{2}}$.

Finally, there is the question of the term $L_b/(N_1\varrho)^{\frac{2}{3}}$ in (105). It is due to this that the field is exponentially damped in the shadow. The replacement must be representative of the whole stretch of boundary that is involved and accordingly we employ

$$\int_A^B N_l \,dl/(N_l\rho_l)^{\frac{2}{3}},$$

where l denotes arc length along the boundary and N_l, ρ_l are values at the point where the arc length is l. Written in this form both numerator and denominator are invariant under a conformal mapping.

With these replacements (105) becomes

$$\psi = \psi_i \frac{e^{\frac{1}{4}\pi i}}{\sqrt{\pi}} \int_{\{k(L-L_i)\}^{\frac{1}{2}}}^{\infty} e^{-i\beta^2} d\beta - \frac{(N_A N_B \rho_A \rho_B)^{\frac{1}{6}}}{(\frac{1}{2}k)^{\frac{1}{6}}} \gamma_A \gamma_B e^{-ikL} G\left\{(\tfrac{1}{2}k)^{\frac{1}{3}} \int_A^B \frac{N_l \,dl}{(N_l\rho_l)^{\frac{2}{3}}}\right\} \tag{107}$$

which is now invariant under a conformal mapping. The function $G(\tau)$ does not involve the medium or boundary except through the constant Z. It is therefore invariant under a conformal mapping. This is another reason for choosing the boundary condition for our model in the form (26). The form which is invariant under a conformal mapping is

$$\frac{\partial \psi}{\partial n} + kN_l\left(\frac{2}{kN_l\rho_l}\right)^{\frac{1}{3}} Z\psi = 0. \tag{108}$$

It may be surmised that the invariant formula analogous to (96) is

$$L - L_i = \frac{1}{2}\left(\frac{\gamma_i L_b}{\gamma_A^2 N_A \rho_A}\right)^2. \tag{109}$$

In view of the invariance of (107) under a conformal mapping it follows that, to find the field in any medium which together with its boundary is obtained by a conformal mapping from our original medium and boundary, it is only necessary to replace ψ, L, N, ρ, γ in (107) by $\psi^{(1)}$, $L^{(1)}$, $N^{(1)}$, $\rho^{(1)}$, $\gamma^{(1)}$. To put it another way, these quantities can be calculated directly in the new medium by geometrical optics and substituted at once in (107). Since Z is a constant G is unaffected. The main difference is that a different boundary condition will hold in the new medium. However, in the cases $Z = 0$ and $Z = \infty$ even the boundary condition is not altered. On account of the large number of different media and boundaries that can be obtained by conformal mapping, the only restriction being that the mapping does not have a singularity in the region under consideration so that the conformal property does not fail, one is naturally led to consider the possibility that (107) is of universal applicability.

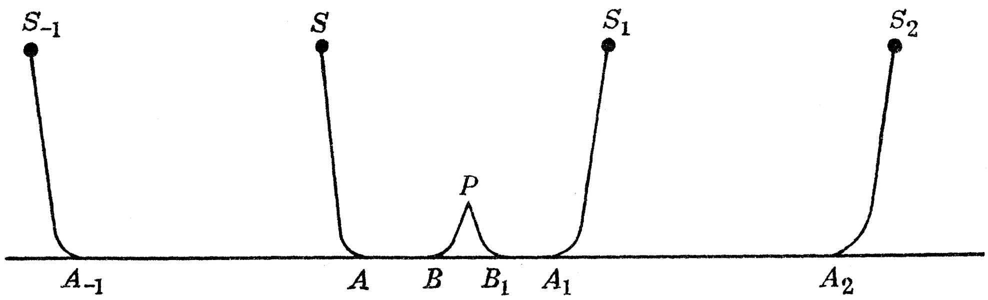

FIGURE 3. S, S_1, S_2, ..., S_{-1} are images of the source in the homogeneous medium and each provides a contribution (e.g. $SAPB$ and $S_2A_2B_1P$) at P.

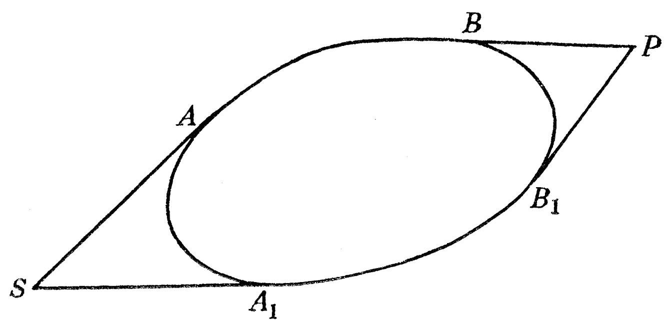

FIGURE 4. There are contributions at P from all paths starting at S and tangential at A, B, B_1 and A_1.

Before attempting confirmation of this consider the problem of a line source in a homogeneous medium outside a finite convex obstacle. Let the potential function V when the boundary is raised to a constant potential be given by

$$W = U + \mathrm{i}V = g(z).$$

Outside a circle which completely encloses the obstacle,

$$W = \mathrm{i}B \ln z + \Sigma a_n/z^n,$$

where B is real and non-zero. Thus U is multiple-valued, increasing by $2\pi B$ in every circuit of the obstacle. If therefore the conformal mapping $W = g(z)$ is made the problem becomes one of an inhomogeneous medium over a plane boundary with a periodic distribution of sources, the separation between them being $2\pi B$ (figure 3). According to the above theory each source will give a contribution of the type (107) at the point of observation P, the points of glancing incidence being (A, B), (A_1, B_1), (A_2, B_1) ... (A_{-1}, B) .... Returning to the original medium the points $A_1, A_2, \ldots$ are superimposed as are the points $A, A_{-1}, \ldots$. On account of the invariance under a conformal mapping this can be interpreted in the original medium as there being a contribution (107) at P from each of the paths

$$AB, \quad ABB_1A_1AB, \quad ABB_1A_1ABB_1A_1AB, \quad \ldots, \quad A_1B_1, \quad A_1B_1BAA_1B_1, \quad \ldots \quad \text{(figure 4)}.$$

Those paths which go completely round the obstacle correspond to creeping waves.

The above theory may be summarized as follows. Away from any boundary the field obeys the standard laws of geometrical optics. Shadow regions and diffraction are caused by behaviour near the boundary and are dictated by the effective curvature $1/\rho$ which is a combination of the curvatures of the boundary and the ray tangent to a boundary. As a model we select a medium with a linear gradient and then cast the field into a form invariant under a conformal mapping. In this form we suggest the field is appropriate to all media and boundaries which satisfy the conditions:

(i) The frequency is high. At the boundary this condition takes the form

$$kN_l\rho_l \gg 1$$

on account of (45). However, *there must not be a point on the boundary at which both the curvatures of the tangent ray and boundary vanish.* This would correspond to $q = 0$ so that all shadow and diffraction effects would disappear from our model, and our formula could no longer be regarded as reliable. In any case the boundary condition (108) at that point would be obliged to be $\partial\psi/\partial n = 0$.

(ii) The shadow boundary is a single ray tangent to the obstacle. The present theory does not apply when the shadow boundary is a caustic of the incident rays.

(iii) The rays do not form a caustic between the source, point of observation and boundary. However, if the caustic were sufficiently far from the boundary it might be possible to predict the field in a region surrounding the boundary by (107) and then extend the calculations by ordinary geometrical optics through the caustic.

(iv) The source and point of observation must not be close to the boundary of the obstacle. For our model the condition is (66). There are several ways of generalizing this but perhaps the most convenient is

$$kL_s^{\frac{3}{2}}/(N_s\rho_s)^{\frac{1}{2}} \gg 1,$$

where L_s is the optical path length of the shortest ray to the boundary from the source (or point of observation) and N_s, ρ_s are values where this ray meets the boundary. Note that the restriction of having the point of observation nearer than the source is no longer necessary since (107) satisfies the reciprocity theorem.

(v) The contributions from all points of glancing incidence must be added. If, by proceeding along the boundary, we return to a point of glancing incidence then an additional contribution is necessary.

Under these conditions (107) should give the field with a relative error of $O(k^{-\frac{2}{3}})$ when there is a line source producing a nearby field of $e^{-ikN_0r}/(N_0r)^{\frac{1}{2}}$ provided that the boundary condition is (108).

A word is necessary about the significance of L_i. This is the optical path length of an incident ray from the source to point of observation assuming that the obstacle is absent.

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There might appear to be a possible difficulty over this definition in the shadow region but, in fact, a short way into the shadow from the shadow boundary the asymptotic formulae (80) and (A 5) apply so that L_i disappears from the field and only the series is left as in (102) and (103). L is the sum of three parts (see figure 2), the optical path length of the ray of glancing incidence from the source, the optical path length along the boundary to the point of tangency of the glancing ray through the point of observation and the optical path length of this latter ray. When the point of observation is in the illuminated region the boundary optical length is taken to be negative. A short distance from the shadow boundary into the illuminated region the precise definition of L is immaterial in the sense that the asymptotic formulae (93) and (A 7) apply and the field is $\psi_i+\psi_r$. In fact the rule is: calculate (107) first near the shadow boundary; in moving away from the boundary as soon as $k(L-L_i)$ is large replace (107) by $\psi_i+\psi_r$ in the illuminated region or by the series in G, as exemplified by (102) and (103), in the shadow region. It must be remembered that $(L-L_i)^{\frac{1}{2}}$ is to be taken as positive or negative according as one is in the shadow or illuminated region. Also A and B are the points of glancing incidence from the source and point of observation respectively.

12. The field on the boundary

As explained in condition (iv) in the preceding section the formula does not apply when the point of observation is near the boundary. Often it is useful to know the field on the boundary. To find this return to the model with the linear gradient and put $y = 0$ in (43), thereby obtaining

$$\psi = \frac{BkN_1}{2\pi}\int_{-\infty}^{\infty} \mathrm{Ai}\,\{\kappa^{\frac{2}{3}}(1+qy_0-\alpha^2)\,\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}\}\,[\mathrm{Ai}\,\{\kappa^{\frac{2}{3}}(1-\alpha^2)\,\mathrm{e}^{\pi\mathrm{i}}\}+R_1(\alpha^2)\,\mathrm{Ai}\,\{\kappa^{\frac{2}{3}}(1-\alpha^2)\,\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}\}]\,\mathrm{e}^{-\mathrm{i}\alpha kN_1x}\,\mathrm{d}\alpha$$

$$= -\frac{\mathrm{i}BkN_1\,\mathrm{e}^{-\frac{1}{3}\pi\mathrm{i}}}{4\pi^2}\int_{-\infty}^{\infty}\frac{\mathrm{Ai}\,\{\kappa^{\frac{2}{3}}(1+qy_0-\alpha^2)\,\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}\}\,\mathrm{e}^{-\mathrm{i}\alpha kN_1x}}{\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}\,\mathrm{Ai}'\,\{\kappa^{\frac{2}{3}}(1-\alpha^2)\,\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}\}+Z\,\mathrm{Ai}\,\{\kappa^{\frac{2}{3}}(1-\alpha^2)\,\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}\}}\,\mathrm{d}\alpha$$

after a use of (57).

Assume again that the source is not near the boundary; the main contribution to ψ near the point of glancing incidence again comes from the neighbourhood of $\alpha = 1$ where the Airy function in the numerator can be replaced by its asymptotic approximation (41). Putting $\alpha = 1+\frac{1}{2}\beta$ and retaining only the first power of β we obtain

$$\psi = \frac{k^{\frac{1}{2}}\,\mathrm{e}^{-\frac{1}{3}\pi\mathrm{i}}}{2^{\frac{3}{2}}\pi\mathrm{i}}\,\kappa^{\frac{1}{6}}\,\frac{\mathrm{e}^{-\mathrm{i}kL}}{(qy_0)^{\frac{1}{4}}}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{-\frac{1}{2}\mathrm{i}kL_b\beta}}{\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}\,\mathrm{Ai}'\,\{\kappa^{\frac{2}{3}}\beta\,\mathrm{e}^{-\frac{2}{3}\pi\mathrm{i}}\}+Z\,\mathrm{Ai}\,\{\kappa^{\frac{2}{3}}\beta\,\mathrm{e}^{-\frac{2}{3}\pi\mathrm{i}}\}}\,\mathrm{d}\beta.$$

From (29) with $A = N_1$, $y = 0$ we have $\gamma_A = (\frac{1}{2}q/N_1)^{\frac{1}{2}}/(qy_0)^{\frac{1}{4}}$ so that the formula for ψ can be written

$$\psi = \gamma_A\,\mathrm{e}^{-\mathrm{i}kL}\,F\{\tfrac{1}{2}kL_b\kappa^{-\frac{2}{3}}\}, \tag{110}$$

where

$$F(\tau) = \frac{\mathrm{e}^{-\frac{1}{3}\pi\mathrm{i}}}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{-\mathrm{i}\mu\tau}}{\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}\,\mathrm{Ai}'\,(\mu\,\mathrm{e}^{-\frac{2}{3}\pi\mathrm{i}})+Z\,\mathrm{Ai}\,(\mu\,\mathrm{e}^{-\frac{2}{3}\pi\mathrm{i}})}\,\mathrm{d}\mu.$$

Various properties of F are derived in appendix B.

If (B 4) is used in (110) for $L_b > 0$ (64) is reproduced. On the other hand if L_b is sufficiently negative for (B 5) to be valid

$$\psi = \frac{kL_b\gamma_A\exp\{-\mathrm{i}kL+\frac{1}{24}\mathrm{i}kL_b^3(q/N_1)^2\}}{kL_b-\mathrm{i}Z\kappa^{\frac{2}{3}}}.$$

Expanding (30) and (31) with $y = 0$, about $A = N_1$ we find

$$L_i = L - \tfrac{1}{24} L_b^3 (q/N_1)^2,$$

$$\cos\chi = -qL_b/2N_1$$

so that
$$\psi = \{1 + R(\chi)\}\psi_i. \tag{111}$$

This is the same as would be predicted by geometrical optics.

Modifying (110) as in the preceding section we take, for the value of ψ at the point C of the boundary,

$$\psi = \gamma_A e^{-ikL} F\left\{(\tfrac{1}{2}k)^{\frac{1}{3}} \int_A^C \frac{N_l \, dl}{(N_l \rho_l)^{\frac{2}{3}}}\right\}. \tag{112}$$

This is, of course, subject to the conditions (i) to (v) of the preceding section except for the alteration in the point of observation. In using (112), as soon as C is sufficiently far from A on the illuminated side for (B 5) to be valid we replace (112) by (111).

Under the same conditions

$$\frac{\partial \psi}{\partial n} = -kN_C \left(\frac{2}{kN_C \rho_C}\right)^{\frac{1}{3}} Z\gamma_A e^{-ikL} F\left\{(\tfrac{1}{2}k)^{\frac{1}{3}} \int_A^C \frac{N_l \, dl}{(N_l \rho_l)^{\frac{2}{3}}}\right\}. \tag{113}$$

By means of the reciprocity theorem the field due to a source on the boundary at a point some distance away from the boundary can be deduced.

13. The stratified medium with monotonically increasing refractive index

In this section we consider the verification of (107) when the boundary is $y = 0$ and the refractive index of the medium in $y > 0$ depends only on y and increases monotonically from N_1 $(\neq 0)$ at $y = 0$ to an infinite value at $y = \infty$. We shall further suppose that the derivative of N is non-zero everywhere; in particular, that the normal derivative N_1' at $y = 0$ is non-zero.

The equation to be satisfied by Ψ is now

$$d^2\Psi/dy^2 + k^2(N^2 - N_1^2\alpha^2)\Psi = 0. \tag{114}$$

If $\alpha \geqslant 1$ there is a $y \geqslant 0$ such that $N^2 = N_1^2\alpha^2$ and only one such y since N increases monotonically with y. For $|\alpha| < 1$, $N^2 > N_1^2\alpha^2$ for $y \geqslant 0$, but it is convenient to imagine N continued below $y = 0$ so that it is still monotonically increasing; there will then be a single negative y for which $N^2 - N_1^2\alpha^2$ vanishes when $|\alpha| < 1$. Denote the simple zero, which now exists for all α, of $N^2 - N_1^2\alpha^2$ by y_1; it will be a function of α.

Make the change of variable

$$\Psi = \left(\frac{\xi}{N^2 - N_1^2\alpha^2}\right)^{\frac{1}{4}} v(\xi),$$

where
$$\tfrac{2}{3}\xi^{\frac{3}{2}} = \int_{y_1}^{y} (N^2 - N_1^2\alpha^2)^{\frac{1}{2}} \, dy \quad (y \geqslant y_1, \ \xi \geqslant 0), \tag{115}$$

$$\tfrac{2}{3}(-\xi)^{\frac{3}{2}} = -\int_{y_1}^{y} (N_1^2\alpha^2 - N^2)^{\frac{1}{2}} \, dy \quad (y \leqslant y_1, \ \xi \leqslant 0). \tag{116}$$

Then (114) transforms into
$$\frac{d^2 v}{d\xi^2} + \left\{k^2\xi + \frac{d^2 g/dy^2}{(d\xi/dy)^2 g}\right\} v = 0, \tag{117}$$

where
$$g = \frac{1}{(d\xi/dy)^{\frac{1}{2}}} = \left(\frac{\xi}{N^2 - N_1^2\alpha^2}\right)^{\frac{1}{4}}.$$

To a first approximation (117) is
$$\frac{d^2v}{d\xi^2}+k^2\xi v=0$$
with an error which is of smaller order than the one made in taking (107) to be the field. Since this equation can be converted to Airy's equation (38) Ψ can be expressed in terms of Airy functions whose arguments are multiples of ξ. For example, with a line source at $(0, y_0)$,
$$\Psi_i=\frac{4.2^{\frac{1}{2}}\pi^{\frac{3}{2}}\,e^{\frac{1}{12}\pi i}}{k^{\frac{1}{6}}}\left(\frac{\xi}{N^2-N_1^2\alpha^2}\right)^{\frac{1}{4}}\left(\frac{\xi_0}{N_0^2-N_1^2\alpha^2}\right)^{\frac{1}{4}}\operatorname{Ai}(k^{\frac{2}{3}}\xi\,e^{\frac{1}{3}\pi i})\operatorname{Ai}(k^{\frac{2}{3}}\xi_0\,e^{\pi i})$$
for $\xi>\xi_0$ or $y>y_0$ where ξ_0 is the value of ξ at $y=y_0$.

The boundary condition (108) becomes, since $\rho=\varrho$,
$$\partial\psi/\partial y+(2k^2N_1N_1')^{\frac{1}{3}}Z\psi=0$$
on $y=0$. Hence the solution to our problem for $\xi<\xi_0$ is
$$\psi=2^{\frac{3}{2}}\pi^{\frac{1}{2}}\,e^{\frac{1}{12}\pi i}k^{\frac{5}{6}}N_1\int_{-\infty}^{\infty}\left(\frac{\xi}{N^2-N_1^2\alpha^2}\right)^{\frac{1}{4}}\left(\frac{\xi_0}{N_0^2-N_1^2\alpha^2}\right)^{\frac{1}{4}}\operatorname{Ai}(k^{\frac{2}{3}}\xi_0\,e^{\frac{1}{3}\pi i})\Big\{\operatorname{Ai}(k^{\frac{2}{3}}\xi\,e^{\pi i})$$
$$-\left[\frac{d\{g\operatorname{Ai}(k^{\frac{2}{3}}\xi\,e^{\pi i})\}/dy+(2k^2N_1N_1')^{\frac{1}{3}}Zg\operatorname{Ai}(k^{\frac{2}{3}}\xi\,e^{\pi i})}{d\{g\operatorname{Ai}(k^{\frac{2}{3}}\xi\,e^{\frac{1}{3}\pi i})\}/dy+(2k^2N_1N_1')^{\frac{1}{3}}Zg\operatorname{Ai}(k^{\frac{2}{3}}\xi\,e^{\frac{1}{3}\pi i})}\right]_{y=0}\operatorname{Ai}(k^{\frac{2}{3}}\xi\,e^{\frac{1}{3}\pi i})\Big\}\,e^{-i\alpha kN_1x}\,d\alpha. \tag{118}$$

Note that if N had been bounded as $y\to\infty$, further consideration would have been necessary of the behaviour at infinity for sufficiently large values of α—when ξ might be imaginary. However, with our assumption ξ is always real at $y\to\infty$.

The discussion of (118) is similar to that of (43) and we shall illustrate only certain salient points. For example, as α increases from zero ξ steadily decreases (for fixed y), passing through a zero to the right of $\alpha=1$. The equation corresponding to the upper sign of (46) is
$$\frac{\partial}{\partial\alpha}\{-\tfrac{2}{3}\xi_0^{\frac{3}{2}}+\tfrac{2}{3}\xi^{\frac{3}{2}}-\alpha N_1x\}=0,$$
or
$$\int_{y_0}^{y}\frac{N_1^2\alpha}{(N^2-N_1^2\alpha^2)^{\frac{1}{2}}}\,dy+N_1x=0. \tag{119}$$

From (13) we see that this is an incident ray with $A=\alpha N_1$ going down from the source. The phase of the contribution from this point of stationary phase is $-ik(\frac{2}{3}\xi^{\frac{3}{2}}-\frac{2}{3}\xi_0^{\frac{3}{2}}-\alpha N_1x)$ with α given by (119). On account of (115) this is $-ikL_i$ where L_i is given by (19). Similarly the amplitude proves to be γ_i as given by (18). Thus the points of stationary phase reproduce geometrical optics as in the case of the linear gradient.

Near the shadow boundary which, according to (22) with $h=0$, has equation
$$x=\int_0^{y}\frac{N_1}{(N^2-N_1^2)^{\frac{1}{2}}}\,dy+\int_0^{y_0}\frac{N_1}{(N^2-N_1^2)^{\frac{1}{2}}}\,dy$$
we need consider only the interval of integration around $\alpha=1$. When $\alpha=1$, $y_1=0$ and when $\alpha^2=1+\beta$, y_1 satisfies
$$N^2(y_1)=N_1^2(1+\beta)$$
or
$$N_1^2+2y_1N_1N_1'=N_1^2(1+\beta)$$
to a first approximation. Since $N_1'\neq 0$,
$$y_1\approx N_1\beta/2N_1' \tag{120}$$

and, in general, y_1 is a power series in β when $\alpha \approx 1$. Thus near $\alpha = 1$ the right-hand side of (115) can be expanded in a Taylor series of powers of β. One finds without difficulty

$$\int_{y_1}^{y} \{N^2 - N_1^2(1+\beta)\}^{\frac{1}{2}} \mathrm{d}y = \int_0^y (N^2 - N_1^2)^{\frac{1}{2}} \mathrm{d}y - \tfrac{1}{2}N_1^2\beta \int_0^y \frac{\mathrm{d}y}{(N^2-N_1^2)^{\frac{1}{2}}}$$
$$+\tfrac{1}{8}N_1^4\beta^2 \left\{\frac{1}{NN'(N^2-N_1^2)^{\frac{1}{2}}} - \int_0^y \frac{1}{(N^2-N_1^2)^{\frac{1}{2}}} \frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{1}{NN'}\right)\mathrm{d}y\right\}.$$

Hence from the interval with α slightly greater than unity the first term of (118) gives (neither the source nor point of observation being near the boundary)

$$\frac{k^{\frac{1}{2}} N_1 \mathrm{e}^{\frac{1}{4}\pi \mathrm{i} - \mathrm{i}kL}}{2^{\frac{3}{2}}\pi^{\frac{1}{2}}(N^2-N_1^2)^{\frac{1}{4}}(N_0^2-N_1^2)^{\frac{1}{4}}} \int_0^\infty \exp\left(-\tfrac{1}{2}\mathrm{i}k\beta L_b - \tfrac{1}{4}\mathrm{i}kL_1\beta^2\right)\mathrm{d}\beta, \tag{121}$$

where $L = \int_0^{y_0} (N^2-N_1^2)^{\frac{1}{2}}\mathrm{d}y + \int_0^y (N^2-N_1^2)^{\frac{1}{2}}\mathrm{d}y + N_1 x,$

$$L_b = N_1 x - \int_0^y \frac{N_1^2}{(N^2-N_1^2)^{\frac{1}{2}}}\mathrm{d}y - \int_0^{y_0} \frac{N_1^2}{(N^2-N_1^2)^{\frac{1}{2}}}\mathrm{d}y,$$

$$L_1 = \frac{N_1^4}{2NN'(N^2-N_1^2)^{\frac{1}{2}}} + \frac{N_1^4}{2N_0N_0'(N_0^2-N_1^2)^{\frac{1}{2}}} - \tfrac{1}{2}N_1^4 \int_0^y \frac{1}{(N^2-N_1^2)^{\frac{1}{2}}}\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{1}{NN'}\right)\mathrm{d}y$$
$$-\tfrac{1}{2}N_1^4 \int_0^{y_0} \frac{1}{(N^2-N_1^2)^{\frac{1}{2}}}\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{1}{NN'}\right)\mathrm{d}y - \tfrac{1}{2}N_1 x.$$

From (14) the optical path length from the source to the point of glancing incidence is

$$-\int_{y_0}^{0} N^2(N^2-N_1^2)^{-\frac{1}{2}}\mathrm{d}y.$$

The optical path length of the ray of glancing incidence through the point of observation is

$$\int_0^y N^2(N^2-N_1^2)^{-\frac{1}{2}}\mathrm{d}y$$

and the abscissa of the point of contact of this ray with the boundary is

$$x - \int_0^y N_1(N^2-N_1^2)^{-\frac{1}{2}}\mathrm{d}y.$$

From (21) the point of contact of the ray of glancing incidence from the source has abscissa

$$-\int_{y_0}^{0} N_1(N^2-N_1^2)^{-\frac{1}{2}}\mathrm{d}y.$$

Consequently, the interpretation of L and L_b is the same as that in figure 2.

The incident ray to the point of observation has optical path length

$$L_i = \int_h^{y_0} N^2(N^2-A^2)^{-\frac{1}{2}}\mathrm{d}y + \int_h^y N^2(N^2-A^2)^{-\frac{1}{2}}\mathrm{d}y$$

and the equation of the rising portion is

$$x = \int_h^y A(N^2-A^2)^{-\frac{1}{2}}\mathrm{d}y + \int_h^{y_0} A(N^2-A^2)^{-\frac{1}{2}}\mathrm{d}y,$$

where A is nearly equal to N_1 and h is nearly zero. By expanding about $A = N_1$ we find

$$L_i = L - \tfrac{1}{4}L_b^2/L_1 \tag{122}$$

which is the same as (96). Hence (121) can be written

$$\frac{N_1\,\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}-\mathrm{i}kL_i}}{2^{\frac{1}{2}}\pi^{\frac{1}{2}}(N^2-N_1^2)^{\frac{1}{4}}(N_0^2-N_1^2)^{\frac{1}{4}}L_1^{\frac{1}{2}}}\int_{\{k(L-L_i)\}^{\frac{1}{2}}}^{\infty}\mathrm{e}^{-\mathrm{i}\beta^2}\,\mathrm{d}\beta.$$

When $A = N_1$ and $h = 0$ in (23)

$$\gamma_i^2 = \frac{N_1^2}{(N^2-N_1^2)^{\frac{1}{2}}(N_0^2-N_1^2)^{\frac{1}{2}}\,2L_1}$$

so that the first term of (107) is reproduced.

For the second term of (118) when $\alpha^2 = 1+\beta$ first consider the bracket $[\,]_{y=0}$. Denote by ξ_1 the value of ξ when $y = 0$. For $\alpha^2 = 1+\beta$ $(\beta \geqslant 0)$, $\xi_1 \leqslant 0$ so that (116) is appropriate. Use of (120) gives

$$\tfrac{2}{3}(-\xi_1)^{\frac{3}{2}} = \int_0^{y_1}(N_1^2\beta-2N_1N_1'y)^{\frac{1}{2}}\,\mathrm{d}y$$
$$= \tfrac{1}{3}N_1^2\beta^{\frac{3}{2}}/N_1'$$

to a first approximation. Thus

$$\xi_1 \approx -\beta(N_1^2/2N_1')^{\frac{2}{3}} \tag{123}$$

and, consequently $[\mathrm{d}\xi/\mathrm{d}y]_{y=0} \approx (2N_1N_1')^{\frac{1}{3}}$. Also

$$\left[\frac{1}{g}\frac{\mathrm{d}g}{\mathrm{d}y}\right]_{y=0} = \left[\frac{1}{4}\frac{1}{\xi}\frac{\mathrm{d}\xi}{\mathrm{d}y}-\frac{1}{2}\frac{NN'}{N^2-N_1^2\alpha^2}\right]_{y=0}$$
$$= \left[\frac{1}{4\xi}\left(\frac{N^2-N_1^2\alpha^2}{\xi}\right)^{\frac{1}{2}}-\frac{1}{2}\frac{NN'}{N^2-N_1^2\alpha^2}\right]_{y=0}$$
$$= \frac{N_1(1-\alpha^2)^{\frac{1}{2}}}{4\xi_1(-\xi_1)^{\frac{1}{2}}}-\frac{N_1'}{2N_1(1-\alpha^2)}.$$

On account of (123) the right-hand side is not singular as $\alpha \to 1$ but $O(1)$. Hence, rejecting a term $O(k^{-\frac{2}{3}})$ we have for the bracket $[\,]_{y=0}$ in (118)

$$\frac{\mathrm{e}^{\pi\mathrm{i}}\,\mathrm{Ai}'\,(k^{\frac{2}{3}}\xi_1\,\mathrm{e}^{\pi\mathrm{i}})+Z\,\mathrm{Ai}\,(k^{\frac{2}{3}}\xi_1\,\mathrm{e}^{\pi\mathrm{i}})}{\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}\,\mathrm{Ai}'\,(k^{\frac{2}{3}}\xi_1\,\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}})+Z\,\mathrm{Ai}\,(k^{\frac{2}{3}}\xi_1\,\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}})}.$$

Therefore the contribution to (118) is

$$-\frac{(2N_1')^{\frac{2}{3}}\,\mathrm{e}^{-\frac{1}{12}\pi\mathrm{i}-\mathrm{i}kL}}{k^{\frac{1}{6}}N_1^{\frac{4}{3}}2(2\pi)^{\frac{1}{2}}(N^2-N_1^2)^{\frac{1}{4}}(N_0^2-N_1^2)^{\frac{1}{4}}}$$
$$\times\int_0^{\infty}\frac{\mathrm{e}^{\pi\mathrm{i}}\,\mathrm{Ai}'\,(\mu)+Z\,\mathrm{Ai}\,(\mu)}{\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}\mathrm{Ai}'\,(\mu\,\mathrm{e}^{-\frac{2}{3}\pi\mathrm{i}})+Z\,\mathrm{Ai}\,(\mu\,\mathrm{e}^{-\frac{2}{3}\pi\mathrm{i}})}\exp\left\{-\tfrac{1}{2}\mathrm{i}kL_b\left(\frac{2N_1'}{kN_1^2}\right)^{\frac{2}{3}}\mu\right\}\mathrm{d}\mu.$$

The contribution from $\alpha^2 < 1$, may be dealt with similarly to give a total contribution of

$$-\frac{(2N_1')^{\frac{2}{3}}\,\mathrm{e}^{-\mathrm{i}kL}}{k^{\frac{1}{6}}N_1^{\frac{4}{3}}2^{\frac{1}{2}}(N^2-N_1^2)^{\frac{1}{4}}(N_0^2-N_1^2)^{\frac{1}{4}}}\,G\left\{\tfrac{1}{2}kL_b\left(\frac{2N_1'}{kN_1^2}\right)^{\frac{2}{3}}\right\}.$$

Now, from (20)

$$\gamma_A^2 = \frac{N_1'/N_1}{(N_0^2-N_1^2)^{\frac{1}{2}}},$$
$$\gamma_B^2 = \frac{N_1'/N_1}{(N^2-N_1^2)^{\frac{1}{2}}},$$
$$\frac{1}{\rho_A} = \frac{1}{\rho_B} = \frac{1}{\varrho} = N_1'/N_1.$$

Consequently the second term of (107) is reproduced. This completes the verification for the horizontally stratified medium with monotonically increasing refractive index. Because of the invariance under conformal mapping (107) will be valid for all problems which can be obtained by conformal mapping.

It is of interest to observe that (122) can be written in the invariant form

$$(L-L_i)^{\frac{1}{2}} = \gamma_i L_b / 2^{\frac{1}{2}} \gamma_A \gamma_B N_A \rho_A \tag{124}$$

as conjectured in (109).

14. The homogeneous medium

In the preceding section (107) was verified for a variable medium with a straight boundary. This section is devoted to a consideration of certain cases when the boundary is curved but the medium homogeneous. There is no loss of generality in taking $N = 1$.

(a) *The circular cylinder*

Let the cylinder be of radius a and let the source S and point of observation P be situated as in figure 5 (we assume $0 \leqslant \theta \leqslant \pi$). Since the rays are straight lines in a homogeneous medium their curvature is zero. The curvature of the cylinder is $1/a$ at every point but, according to the rule of § 11, it is to be counted negative, since the centre of curvature is on the opposite side of the boundary to the medium. Hence $\rho_l = a$ and our theory should be applicable for $ka \gg 1$.

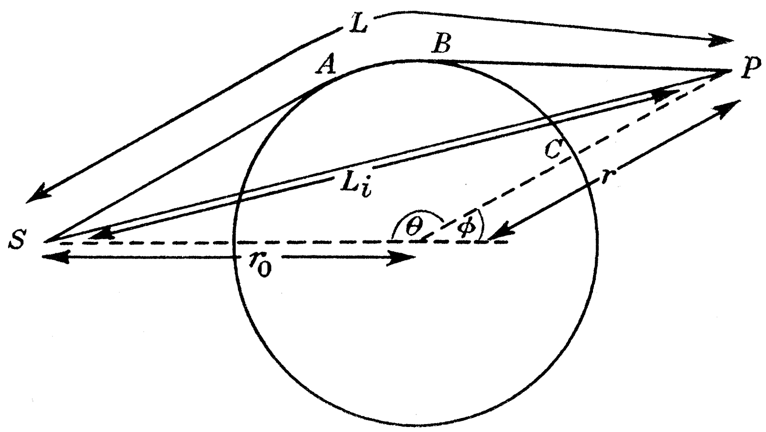

Figure 5. The parameters for a circular cylinder in a homogeneous medium.

With each of the two points of glancing incidence is associated a single ray which constitutes the shadow boundary. Condition (ii) is complied with. Clearly condition (iii) is met. Condition (iv) requires

$$k(r_0-a)^{\frac{3}{2}}/a^{\frac{1}{2}} \gg 1, \quad k(r-a)^{\frac{3}{2}}/a^{\frac{1}{2}} \gg 1. \tag{125}$$

Consider first the upper point of glancing incidence A. The various quantities occurring in (107) are

$$L_i = (r_0^2 + r^2 - 2r_0 r \cos\theta)^{\frac{1}{2}},$$

$$L = (r_0^2 - a^2)^{\frac{1}{2}} + (r^2 - a^2)^{\frac{1}{2}} + a\left(\theta - \cos^{-1}\frac{a}{r_0} - \cos^{-1}\frac{a}{r}\right),$$

$$\psi_i = \mathrm{e}^{-\mathrm{i}kL_i}/L_i^{\frac{1}{2}},$$

$$\gamma_A = (r_0^2 - a^2)^{-\frac{1}{4}},$$

$$\gamma_B = (r^2 - a^2)^{-\frac{1}{4}},$$

$$\int_A^B \frac{N_l \mathrm{d}l}{(N_l \rho_l)^{\frac{2}{3}}} = \int_A^B a^{\frac{1}{3}} \mathrm{d}\theta = a^{\frac{1}{3}}\left(\theta - \cos^{-1}\frac{a}{r_0} - \cos^{-1}\frac{a}{r}\right)$$

where the $\cos^{-1}$ lies in $(0, \frac{1}{2}\pi)$. Substitution in (107) gives

$$I(\chi) = \frac{e^{-ikL_i+\frac{1}{4}\pi i}}{(\pi L_i)^{\frac{1}{2}}} \int_{\{k(L-L_i)\}^{\frac{1}{2}}}^{\infty} e^{-i\beta^2}\, d\beta - \frac{a^{\frac{1}{3}} e^{-ikL}\, G\{(\frac{1}{2}ka)^{\frac{1}{3}}\chi\}}{(\frac{1}{2}k)^{\frac{1}{6}} (r_0^2-a^2)^{\frac{1}{4}} (r^2-a^2)^{\frac{1}{4}}}$$

where $$\chi = \theta - \cos^{-1}\frac{a}{r_0} - \cos^{-1}\frac{a}{r}.$$

Note that L depends on χ but not L_i.

To comply with condition (v) we must consider the possibility of going completely round the circle from A back to A and then going to P from B. Such a process gives a contribution $I(\chi+2n\pi)$ where n is the number of times the circle has been described.

Similarly there is a contribution from the lower point of glancing incidence of

$$I(\chi+2\pi-2\theta),$$

plus those due to circuits of the cylinder.

Hence, the field which satisfies the boundary condition

$$\partial\psi/\partial r + k^{\frac{2}{3}}(2/a)^{\frac{1}{3}} Z\psi = 0$$

on $r = a$ is $$\psi = \sum_{n=0}^{\infty} \{I(\chi+2n\pi) + I(\chi+2\pi-2\theta+2n\pi)\}. \tag{126}$$

Since $\chi+2n\pi$ $(n \geqslant 1)$ is never less than π both $k(L-L_i)$ and $(\frac{1}{2}ka)^{\frac{1}{3}}(\chi+2n\pi)$ are large and positive for these values of n. Therefore asymptotic formulae, as for (103), are appropriate and $I(\chi+2n\pi) = O(\exp\{i\delta_1 e^{-\frac{1}{3}\pi i}(\frac{1}{2}ka)^{\frac{1}{3}}(2n-1)\pi\})$. Such a term can be neglected and so can terms such as $I(\chi+2\pi-2\theta+2n\pi)$. With this simplification

$$\psi = I(\chi) + I(\chi+2\pi-2\theta). \tag{127}$$

Since the problem of the circular cylinder has been solved exactly (126) and (127) can be verified from this solution. Such a verification is not strictly necessary since the mapping $w = i\ln z$ converts the problem into one of a stratified medium with $N^2 = e^{2v}$ above the straight boundary $v = \ln a$. Such a medium comes within the theory of the preceding section and so (126) is valid by the invariant property.

The point of observation is near a shadow boundary when $\chi \approx 0$. For more negative values of χ, $I(\chi)$ gives the field of geometrical optics: $I(\chi+2\pi-2\theta)$ will be exponentially damped and may be neglected unless $r_0 \gg a$. For more positive values of χ, both $I(\chi)$ and $I(\chi+2\pi-2\theta)$ are exponentially damped so that the field is exponentially small behind the cylinder (unless $r_0 \gg a$). Near the shadow boundary L_i can be expanded in powers of χ, namely

$$L_i^2 \approx \frac{(r_0^2-a^2)(r^2-a^2)}{R^2} + 2a\chi\frac{(r_0^2-a^2)^{\frac{1}{2}}(r^2-a^2)^{\frac{1}{2}}}{R} + \chi^2\{a^2-(r_0^2-a^2)^{\frac{1}{2}}(r^2-a^2)^{\frac{1}{2}}\},$$

where $$R = (r_0^2-a^2)^{\frac{1}{2}}(r^2-a^2)^{\frac{1}{2}}/\{(r_0^2-a^2)^{\frac{1}{2}}+(r^2-a^2)^{\frac{1}{2}}\}.$$

Thus $$L_i \approx \frac{(r_0^2-a^2)^{\frac{1}{2}}(r^2-a^2)^{\frac{1}{2}}}{R} + a\chi - \tfrac{1}{2}\chi^2 R,$$

$$(L-L_i)^{\frac{1}{2}} = (\tfrac{1}{2}R)^{\frac{1}{2}}\chi \tag{128}$$

in agreement with (124). These formulae permit the calculation of $I(\chi)$ without the intervention of θ.

The field due to an incident plane wave can be calculated by a direct application of (107) or by taking the preceding results with $r_0 \gg a$ and multiplying by $r_0^{\frac{1}{2}} e^{ikr_0}$. Thus the incident plane wave $e^{-ikr\cos\phi}$ ($\phi = \pi - \theta$) produces the field

$$\psi = I_1(\chi) + I_1(\chi + 2\phi),$$

where

$$I_1(\chi) = \frac{\exp(-ikr\cos\phi + \frac{1}{4}\pi i)}{\pi^{\frac{1}{2}}} \int_{\{k(L-L_i)\}^{\frac{1}{2}}}^{\infty} e^{-i\beta^2} d\beta - \frac{a^{\frac{1}{3}} \exp[-ik\{(r^2-a^2)^{\frac{1}{2}} + a\chi\}]}{(\frac{1}{2}k)^{\frac{1}{6}} (r^2-a^2)^{\frac{1}{4}}} G\{(\tfrac{1}{2}ka)^{\frac{1}{3}}\chi\},$$

$$\chi = \tfrac{1}{2}\pi - \phi - \cos^{-1} a/r,$$

$$L - L_i = (r^2-a^2)^{\frac{1}{2}} + a\chi - r\cos\phi.$$

For fixed non-zero ϕ as r increases a region is always reached in which geometrical optics is valid. Therefore, for $r \gg a$, most interest centres around the region in which ϕ is small and (128) holds. Then

$$\psi \sim e^{-ikr\cos\phi}\left\{1 - \frac{e^{\frac{1}{4}\pi i}}{\sqrt{\pi}} \int_{-(\frac{1}{2}kr)^{\frac{1}{2}}\chi}^{(\frac{1}{2}kr)^{\frac{1}{2}}(\chi+2\phi)} e^{-i\beta^2} d\beta\right\} - \frac{a^{\frac{1}{3}} e^{-ikr}}{(\frac{1}{2}k)^{\frac{1}{6}} r^{\frac{1}{2}}}$$
$$\times [e^{-ika\chi} G\{(\tfrac{1}{2}ka)^{\frac{1}{3}}\chi\} + e^{-ika(\chi+2\phi)} G\{(\tfrac{1}{2}ka)^{\frac{1}{3}}(\chi+2\phi)\}].$$

Since $\chi \sim a/r - \phi$ the second term can be written

$$\int_{-a/r}^{a/r} \exp\{-\tfrac{1}{2}ikr(\phi^2 + 2\phi u + u^2)\} (\tfrac{1}{2}kr)^{\frac{1}{2}} du = \left(\frac{2}{kr}\right)^{\frac{1}{2}} \frac{\sin ka\phi}{\phi} e^{-\frac{1}{2}ikr\phi^2}$$

since the u^2 in the exponent can be neglected. Hence

$$\psi \sim e^{-ikr\cos\phi} - \left(\frac{2}{\pi kr}\right)^{\frac{1}{2}} e^{-ikr + \frac{1}{4}\pi i}$$
$$\times \left[\frac{\sin ka\phi}{\phi} + (\tfrac{1}{2}ka)^{\frac{1}{3}} \pi^{\frac{1}{2}} e^{-\frac{1}{4}\pi i}(e^{-ika\phi} G\{(\tfrac{1}{2}ka)^{\frac{1}{3}}\phi\} + e^{ika\phi} G\{-(\tfrac{1}{2}ka)^{\frac{1}{3}}\phi\})\right]$$

a result which can be obtained independently (see, for example, Jones 1962).

It follows that the sum of the scattering and absorption coefficients (Jones 1955) is

$$\frac{2}{ka}\mathscr{R}[ka + (\tfrac{1}{2}ka)^{\frac{1}{3}} \pi^{\frac{1}{2}} e^{-\frac{1}{4}\pi i} 2G(0)] = 2 + \frac{2^{\frac{5}{3}}\pi^{\frac{1}{2}}}{(ka)^{\frac{2}{3}}} \mathscr{R}\, e^{-\frac{1}{4}\pi i} G(0).$$

For the boundary conditions $Z = 0$ and $Z = \infty$ this can be evaluated by (A 10) and (A 11). For other values of Z see the graphs referred to in appendix A.

Finally, note that (112) gives the value of ψ on the boundary at the point C of figure 5 as

$$\sum_{n=0}^{\infty} \{J(\chi + 2n\pi) + J(\chi + 2\pi - 2\theta + 2n\pi)\},$$

where
$$J(\chi) = \frac{1}{(r_0^2 - a^2)^{\frac{1}{4}}} e^{-ik\{(r_0^2-a^2)^{\frac{1}{2}} + a\chi\}} F\{(\tfrac{1}{2}ka)^{\frac{1}{3}}\chi\}$$

and now
$$\chi = \theta - \cos^{-1}(a/r_0).$$

For the incident plane wave replace J by J_1 where

$$J_1(\chi) = e^{-ika\chi} F\{(\tfrac{1}{2}ka)\}^{\frac{1}{3}}\chi\},$$
$$\chi = \theta - \tfrac{1}{2}\pi = \tfrac{1}{2}\pi - \phi.$$

In general only the terms $n = 0$ of the series are necessary.

(*b*) *The parabolic cylinder*

Let the equation of the parabolic cylinder be

$$y^2 = \xi_0^4 - 2\xi_0^2 x$$

and let the source S be specified by the inclination ν_0 of the tangent to the parabola to the positive x axis and the distance d_0 of S from the point of contact (figure 6). Let the position of the point of observation P be similarly specified. The radius of curvature of the parabola at the point where the tangent is inclined at ν to the positive x axis is $\xi_0^2 \operatorname{cosec}^3 \nu$ so that $\rho = \xi_0^2 \operatorname{cosec}^3 \nu$. Hence our theory should be valid when $k\xi_0^2 \gg 1$.

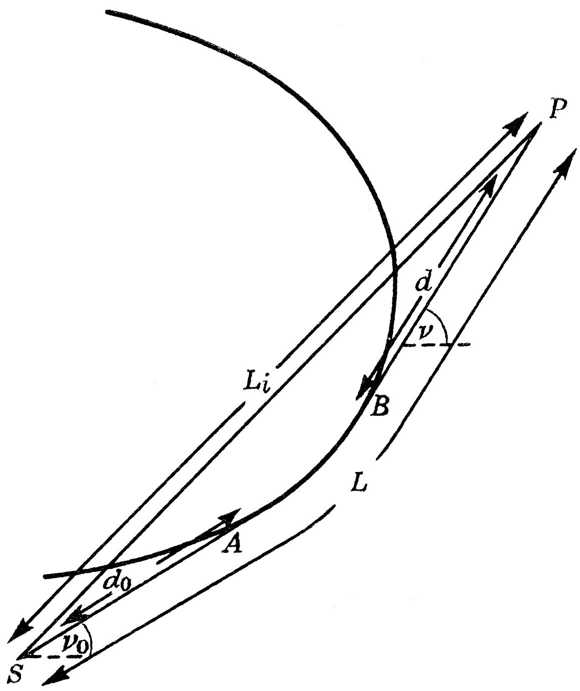

FIGURE 6. The parameters for a parabolic cylinder in a homogeneous medium.

The elementary arc length on the parabola can be calculated from its being $\varrho_1 \mathrm{d}\phi$ and is $\xi_0^2 \operatorname{cosec}^3 \nu \, \mathrm{d}\nu$. Hence

$$L = d_0 + d + \int_{\nu_0}^{\nu} \xi_0^2 \operatorname{cosec}^3 \nu \, \mathrm{d}\nu$$

$$= d_0 + d + \tfrac{1}{2}\xi_0^2 \left\{ \ln \frac{\tan \frac{1}{2}\nu}{\tan \frac{1}{2}\nu_0} - \frac{\cos \nu}{\sin^2 \nu} + \frac{\cos \nu_0}{\sin^2 \nu_0} \right\},$$

$$\gamma_A = d_0^{-\frac{1}{2}}, \quad \gamma_B = d^{-\frac{1}{2}},$$

$$\int_A^B \frac{N_l \, \mathrm{d}l}{(N_l \rho_l)^{\frac{2}{3}}} = \int_{\nu_0}^{\nu} \xi_0^{\frac{2}{3}} \operatorname{cosec} \nu \, \mathrm{d}\nu = \xi_0^{\frac{2}{3}} \ln \frac{\tan \frac{1}{2}\nu}{\tan \frac{1}{2}\nu_0}.$$

L_i is just the distance between S and P. Substitution in (107) now gives the field at P, there being only one point of glancing incidence and one boundary ray to consider. The polar co-ordinates (r, ϕ) of P with respect to the origin are related to d and ν by

$$r \cos \phi - \tfrac{1}{2}\xi_0^2 (1 - \cot^2 \nu) = d \cos \nu,$$

$$r \sin \phi + \xi_0^2 \cot \nu = d \sin \nu.$$

The field for an incident plane wave can be deduced. With the point of observation a large distance from the origin and near the shadow boundary, a direct derivation or (124) gives

$$(L-L_i)^{\frac{1}{2}} = (\tfrac{1}{2}r)^{\frac{1}{2}}(\nu-\phi_0) = (\tfrac{1}{2}r)^{\frac{1}{2}}(\phi-\phi_0+\xi_0^2/2r\sin\phi_0),$$

$$\int_A^B \frac{N_l\,\mathrm{d}l}{(N_l\rho_l)^{\frac{2}{3}}} = \frac{\phi-\phi_0}{\sin\phi_0}$$

for the incident plane wave $\psi_i = \mathrm{e}^{-\mathrm{i}kr\cos(\phi-\phi_0)}$ $(0<\phi_0\leqslant\frac{1}{2}\pi)$. Since the radius of curvature ϱ_0 at the point of glancing incidence is $\xi_0^2\operatorname{cosec}^3\phi_0$, the distant field near the shadow boundary is

$$\psi = \frac{\exp\{-\mathrm{i}kr\cos(\phi-\phi_0)+\frac{1}{4}\pi\mathrm{i}\}}{\sqrt{\pi}}\int_{(\frac{1}{2}kr)^{\frac{1}{2}}(\phi-\phi_0)}^{\infty}\mathrm{e}^{-\mathrm{i}\beta^2}\,\mathrm{d}\beta$$
$$-\left(\frac{2}{\pi kr}\right)^{\frac{1}{2}}\mathrm{e}^{-\mathrm{i}kr+\frac{1}{4}\pi\mathrm{i}}\left[\frac{1-\exp\{-\frac{1}{2}\mathrm{i}k\xi_0^2(\phi-\phi_0)\operatorname{cosec}\phi_0\}}{2\mathrm{i}(\phi-\phi_0)}\right.$$
$$\left.+(\tfrac{1}{2}k\varrho_0)^{\frac{1}{3}}\pi^{\frac{1}{2}}\exp\{-\tfrac{1}{4}\pi\mathrm{i}-\tfrac{1}{2}\mathrm{i}k\xi_0^2(\phi-\phi_0)\operatorname{cosec}\phi_0\}\,G\{(\tfrac{1}{2}k\varrho_0)^{\frac{1}{3}}(\phi-\phi_0)\}\right].$$

These results can be confirmed by independent calculation (see, for example, Rice (1954) and Keller (1956)). Note that in this case the quantity ρ_l in (107) and in the boundary condition (108) is not constant but varies from point to point. This is the first confirmation of the correctness of the forms of the factor involving ρ_l.

The conformal mapping $w = \mathrm{i}(2z)^{\frac{1}{2}}$ where $(2z)^{\frac{1}{2}} = \sqrt{2}$ when $z = 1$ converts the problem into one of a straight boundary $v = \xi_0$ with a medium, in which $N^2 = u^2+v^2$, above. Since such a medium is not included under the theory of § 13, an extension of the domain of validity of (107) has been obtained.

(c) *The arbitrary convex obstacle*

It is possible to write the field which (107) predicts for any convex obstacle, but there is little virtue in so doing because an explicit expression for the integral in the argument of G cannot be obtained in general. Furthermore, there are no exact solutions available for comparison except for the elliptic cylinder. Such results as are known for the elliptic cylinder are in agreement with (107).

It should be remarked that, for plane wave incidence, the sum of the scattering and absorption coefficients predicted by (107) is

$$2+\frac{2^{\frac{5}{3}}\pi^{\frac{1}{2}}}{k^{\frac{2}{3}}D}(\varrho_1^{\frac{1}{3}}+\varrho_2^{\frac{1}{3}})\,\mathscr{R}\,\mathrm{e}^{-\frac{1}{4}\pi\mathrm{i}}\,G(0),$$

where D is the length of wave front intercepted by the obstacle and ϱ_1, ϱ_2 are its finite radii of curvature at the points of glancing incidence.

15. The circular cylinder in a radially stratified medium

Consider the problem of a circular cylinder of radius a surrounded by a medium in which the refractive index varies radially but is independent of angle. Suppose that the refractive index increases monotonically from N_1 at the cylinder to N_∞ at infinity. The mapping $w = \mathrm{i}\ln z$ converts the medium into one of the type discussed in § 13 so that (107) is known to apply. No additional information about the range of validity of (107) is therefore forthcoming but it seems worthwhile to give the formulae predicted in view of the practical importance of the problem.

First, it is necessary to determine the rays. In vector form (8) are

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(N\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}\right) = \operatorname{grad} N.$$

Taking the transverse component of this equation we obtain

$$N\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}s}\left(r^2\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)+\frac{\mathrm{d}N}{\mathrm{d}s}r\frac{\mathrm{d}\theta}{\mathrm{d}s} = 0$$

since N does not depend on θ. Here r and θ are polar co-ordinates with the centre of the cylinder as pole. Hence

$$Nr^2\frac{\mathrm{d}\theta}{\mathrm{d}s} = \text{constant} = A.$$

Therefore

$$\frac{\mathrm{d}r}{\mathrm{d}s} = \pm\left(1-\frac{A^2}{N^2r^2}\right)^{\frac{1}{2}}.$$

and the equation of the rays through (r_0, θ_0) is

$$\theta-\theta_0 = \pm\int_{r_0}^{r}\frac{A\,\mathrm{d}r}{r(N^2r^2-A^2)^{\frac{1}{2}}}.$$

The lower sign corresponds to an incoming ray, the upper to an outgoing ray.

An incoming ray turns at $r = c$ if

$$N(c)\,c = A$$

which occurs if $N_0 r_0 > A \geqslant N_1 a$. The value θ_c at the point of turning is given by

$$\theta_c = -\int_{r_0}^{c}\frac{A\,\mathrm{d}r}{r(N^2r^2-A^2)^{\frac{1}{2}}} \tag{129}$$

when $\theta_0 = 0$, and the ray leaving this point has equation

$$\theta-\theta_c = \int_{c}^{r}\frac{A\,\mathrm{d}r}{r(N^2r^2-A^2)^{\frac{1}{2}}}. \tag{130}$$

Thus the equation of the shadow boundary (figure 7) for a point source at $(r_0, 0)$ is

$$\theta = \int_{a}^{r_0}\frac{N_1 a\,\mathrm{d}r}{r(N^2r^2-N_1^2a^2)^{\frac{1}{2}}}+\int_{a}^{r}\frac{N_1 a\,\mathrm{d}r}{r(N^2r^2-N_1^2a^2)^{\frac{1}{2}}}.$$

The optical path length of a ray from the source after it has turned is given by

$$L_i = \int_{c}^{r_0}\frac{N^2r\,\mathrm{d}r}{(N^2r^2-A^2)^{\frac{1}{2}}}+\int_{c}^{r}\frac{N^2r\,\mathrm{d}r}{(N^2r^2-A^2)^{\frac{1}{2}}}. \tag{131}$$

It follows from (129) and (130) that

$$L_b = N_1 a\left\{\theta-\int_{a}^{r}\frac{N_1 a\,\mathrm{d}r}{r(N^2r^2-N_1^2a^2)^{\frac{1}{2}}}-\int_{a}^{r_0}\frac{N_1 a}{r(N^2r^2-N_1^2a^2)^{\frac{1}{2}}}\mathrm{d}r\right\}, \tag{132}$$

$$L = N_1 a\theta+\int_{a}^{r}(N^2r^2-N_1^2a^2)^{\frac{1}{2}}\,\mathrm{d}r/r+\int_{a}^{r_0}(N^2r^2-N_1^2a^2)^{\frac{1}{2}}\,\mathrm{d}r/r. \tag{133}$$

The calculation of the amplitude proceeds as in § 2 and we find, on an incoming ray,

$$1/\gamma_i^2 = (N_0^2 r_0^2 - A^2)^{\frac{1}{2}} (N^2 r^2 - A^2)^{\frac{1}{2}} \left| \int_{r_0}^{r} N^2 r (N^2 r^2 - A^2)^{-\frac{3}{2}} \, \mathrm{d}r \right|.$$

At a point of turning $\qquad \gamma_c^2 = (1 + cN_c'/N_c)/(N_0^2 r_0^2 - N_c^2 c^2)^{\frac{1}{2}}$

and on a ray after it has turned

$$1/\gamma_i^2 = (N_0^2 r_0^2 - A^2)^{\frac{1}{2}} (N^2 r^2 - A^2)^{\frac{1}{2}} \, |\partial f/\partial A|, \tag{134}$$

where

$$\frac{\partial f}{\partial A} = \frac{2A^2 - N^2 r^2}{(N^2 r^2 - A^2)^{\frac{1}{2}} r^2 N (N' r + N)} + \frac{2A^2 - N_0^2 r_0^2}{(N_0^2 r_0^2 - A^2)^{\frac{1}{2}} r_0^2 N_0 (N_0' r_0 + N_0)}$$
$$+ \int_{r_0}^{c} \frac{2A^2 - N^2 r^2}{(N^2 r^2 - A^2)^{\frac{1}{2}}} \frac{\mathrm{d}}{\mathrm{d}r} \left\{ \frac{1}{Nr^2 (N'r + N)} \right\} \mathrm{d}r + \int_{r}^{c} \frac{2A^2 - N^2 r^2}{(N^2 r^2 - A^2)^{\frac{1}{2}}} \frac{\mathrm{d}}{\mathrm{d}r} \left\{ \frac{1}{Nr^2 (N'r + N)} \right\} \mathrm{d}r.$$

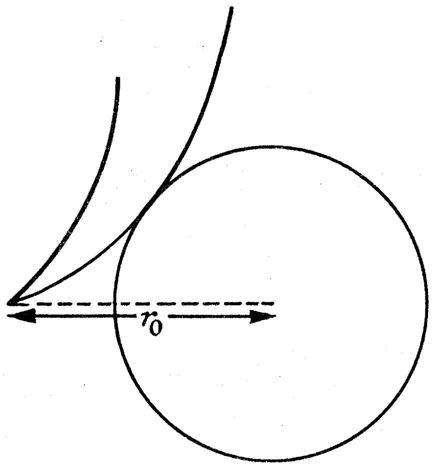

FIGURE 7. The rays of geometrical optics in a radially stratified medium.

It follows that

$$\gamma_A^2 = \frac{1 + aN_1'/N_1}{(N_0^2 r_0^2 - N_1^2 a^2)^{\frac{1}{2}}}, \quad \gamma_B^2 = \frac{1 + aN_1'/N_1}{(N^2 r^2 - N_1^2 a^2)^{\frac{1}{2}}}.$$

Also, from (106) and (104), $\qquad \dfrac{1}{\rho} = \dfrac{N_1'}{N_1} + \dfrac{1}{a}.$

Hence, for the transition across the shadow boundary (107) gives

$$\psi = \frac{\gamma_i \mathrm{e}^{-\mathrm{i}kL_i + \frac{1}{4}\pi \mathrm{i}}}{\pi^{\frac{1}{2}}} \int_{\{k(L-L_i)\}^{\frac{1}{2}}}^{\infty} \mathrm{e}^{-\mathrm{i}\beta^2} \, \mathrm{d}\beta - \frac{\{N_1 a (1 + aN_1'/N_1)^2\}^{\frac{1}{3}} \mathrm{e}^{-\mathrm{i}kL} G[(\frac{1}{2}k)^{\frac{1}{3}} L_b \{(1 + aN_1'/N_1)/N_1 a\}^{\frac{2}{3}}]}{(\frac{1}{2}k)^{\frac{1}{6}} (N_0^2 r_0^2 - N_1^2 a^2)^{\frac{1}{4}} (N^2 r^2 - N_1^2 a^2)^{\frac{1}{4}}}, \tag{135}$$

where γ_i, L, L_i and L_b are given by (134), (133), (131) and (132), respectively. The boundary condition is

$$\frac{\partial \psi}{\partial r} + kN_1 \left\{ \frac{2}{kN_1 a} (1 + aN_1'/N_1) \right\}^{\frac{1}{3}} Z\psi = 0.$$

For the complete field a parallel contribution from the other point of glancing incidence must be added as well as exponentially small terms from the rays which make complete circuits of the cylinder.

Naturally, putting $N = 1$ for all r recovers the formula for a circular cylinder in a homogeneous medium.

Suppose now that the source and point of observation are close enough to the cylinder for N to be replaced by the linear approximation

$$N = 1 + N_1'(r-a)$$

in the region under consideration. Let $r = a+h$, $r_0 = a+h_0$ where $h \ll a$, $h_0 \ll a$. It is possible to meet these conditions and still comply with conditions of the type (125) because the latter require only $h/a \gg (ka)^{-\frac{2}{3}}$. Then

$$N_0^2 r_0^2 - a^2 \approx 2h_0 a(1+N_1' a)$$
$$\approx 2h_0 a^2/a_e,$$

where
$$a_e = \frac{a}{1+N_1' a}.$$

To the same degree of approximation

$$L = a\theta + (2/a_e)^{\frac{1}{2}} \tfrac{2}{3}(h^{\frac{3}{2}} + h_0^{\frac{3}{2}}),$$
$$L_b = a\theta - (\tfrac{1}{2}a_e)^{\frac{1}{2}} 2(h^{\frac{1}{2}} + h_0^{\frac{1}{2}})$$

and near the shadow boundary

$$1/\gamma_i^2 = (2a_e)^{\frac{1}{2}} (h^{\frac{1}{2}} + h_0^{\frac{1}{2}}) = L_i.$$

If $a\theta$ were replaced by $a_e\theta$ these formulae would be the same as those for a circular cylinder of radius a_e in a *homogeneous* medium. In other words, under the conditions stated *the propagation occurs as if it were in a homogeneous medium but the cylinder has an effective radius* a_e provided that the actual distance $a\theta$ along the cylinder surface between source and point of observation is used instead of the effective distance $a_e\theta$. This result has been proved before with special forms of the refractive index such as $N^2 = 1-\eta+\eta a^2/r^2$ (see, for example, Bremmer 1949).

Note also that the formulae are the same as those for the horizontally stratified medium of § 10 provided that the intepretation $x = a\theta$, $y = h$, $y_0 = h_0$, $q = 2/a_e$ is employed.

16. The time-dependent field

This section is concerned with the application of our theory to the equation

$$\nabla^2\Phi - N^2\, \partial^2\Phi/\partial t^2 = 0,$$

where t is time. The source is quiescent until $t = 0$ when it is suddenly switched on and produces a field $(t^2 - N_0^2 r^2)^{-\frac{1}{2}} \mathrm{H}(t - N_0 r)$ at nearby points. Here r is the distance from the source and $\mathrm{H}(x)$ is the Heaviside unit function which is unity for $x > 0$ and zero for $x < 0$.

Let ψ be the Laplace transform of Φ defined by

$$\psi = \int_0^\infty \Phi\, \mathrm{e}^{-st}\, \mathrm{d}t$$

for $\mathscr{R}(s)$ sufficiently large. Then ψ satisfies

$$\nabla^2\psi - s^2 N^2 \psi = 0 \tag{136}$$

and near the source behaves like

$$\int_{N_0 r}^\infty \frac{\mathrm{e}^{-st}}{(t^2 - N_0^2 r^2)^{\frac{1}{2}}}\, \mathrm{d}t = K_0(sN_0 r),$$

K_0 being the modified Bessel function. For large $|s|$ and non-zero r the asymptotic formula for the Bessel function can be employed, so that ψ behaves as $(\pi/2sN_0 r)^{\frac{1}{2}} e^{-sN_0 r}$ near the source.

The substitution $s = \mathrm{i}k$ converts (136) to (1) and produces a source behaviour which is the same as that which has been considered in the preceding sections apart from multiplication by the quantity $(\pi/2s)^{\frac{1}{2}}$. Our formulae have been derived on the assumption that k is real and positive. Now assume that they remain valid when $k = -\mathrm{i}s$. Then, for large $\mathscr{R}(s)$, (107) gives

$$\psi = \left(\frac{\pi}{2s}\right)^{\frac{1}{2}} \left[\frac{\gamma_i e^{-sL_i}}{\pi^{\frac{1}{2}}} \int_{\{s(L-L_i)\}^{\frac{1}{2}}}^{\infty} e^{-\beta^2} d\beta - \frac{(N_A N_B \rho_A \rho_B)^{\frac{1}{6}}}{(\frac{1}{2}s)^{\frac{1}{6}} e^{-\frac{1}{12}\pi i}} \gamma_A \gamma_B e^{-sL} G\left\{(\tfrac{1}{2}s)^{\frac{1}{3}} e^{-\frac{1}{6}\pi i} \int_A^B \frac{N_l \, dl}{(N_l \rho_l)^{\frac{2}{3}}}\right\}\right]. \tag{137}$$

In (137) s does not occur in any factor in which it is not explicitly displayed. The function G is defined, not by (A 1) but by (A 2) to avoid convergence difficulties. If $\mathscr{I}(\tau e^{\frac{1}{3}\pi i}) \geqslant 0$ formula (A 3) can be deduced whereas if $\mathscr{I}(\tau) > 0$ and $\mathscr{I}(\tau e^{\frac{1}{3}\pi i}) < 0$ (A 4) holds.

The field Φ is now found by using the inversion formula

$$\Phi = \frac{1}{2\pi i} \int_{\lambda - i\infty}^{\lambda + i\infty} \psi \, e^{st} \, ds,$$

where λ is a sufficiently large positive real number.

Now $$\frac{1}{2\pi i} \int_{\lambda - i\infty}^{\lambda + i\infty} \frac{e^{st}}{s^{\frac{1}{2}}} \int_{as^{\frac{1}{2}}}^{\infty} e^{-\beta^2} d\beta \, ds = \frac{1}{2t^{\frac{1}{2}}} [H(t^{\frac{1}{2}} - a) + H(-t^{\frac{1}{2}} - a)] \, H(t).$$

Also, if $\mathscr{R}(b e^{\frac{1}{6}\pi i}) \geqslant 0$ and $\mathscr{R}(b e^{-\frac{1}{6}\pi i}) \geqslant 0$,

$$\frac{1}{2\pi i} \int_{\lambda - i\infty}^{\lambda + i\infty} \frac{\exp(st - bs^{\frac{1}{3}})}{s^{\frac{2}{3}}} ds = \frac{3^{\frac{2}{3}} H(t)}{2\pi i t^{\frac{1}{3}}} \int_{\infty \exp(-\frac{1}{6}\pi i)}^{\infty \exp(\frac{1}{6}\pi i)} \exp\{\tfrac{1}{3}s^3 - bs/(3t)^{\frac{1}{3}}\} ds = 3^{\frac{2}{3}} t^{-\frac{1}{3}} H(t) \, \mathrm{Ai}\{b/(3t)^{\frac{1}{3}}\}$$

from (39). In the inversion of the term involving G the exponent satisfies the conditions on b when G is given by (A 2) provided that $\int_A^B \geqslant 0$. When $\int_A^B < 0$ an exponent which has the properties of b can be obtained by deforming the contour of the first integral in (A 2) into the ray $e^{\frac{1}{3}\pi i}$ and of the second integral into the ray $e^{-\frac{1}{3}\pi i}$. When this is done it is found that

$$\Phi = \frac{\gamma_i}{2^{\frac{3}{2}}(t - L_i)^{\frac{1}{2}}} [H\{t^{\frac{1}{2}} - (L - L_i)^{\frac{1}{2}}\} + H\{-t^{\frac{1}{2}} - (L - L_i)^{\frac{1}{2}}\}] \, H(t - L_i)$$
$$- \frac{(N_A N_B \rho_A \rho_B)^{\frac{1}{6}} \gamma_A \gamma_B \pi^{\frac{1}{2}} 3^{\frac{2}{3}}}{(\frac{1}{2})^{\frac{1}{6}} 2^{\frac{1}{2}} e^{-\frac{1}{12}\pi i} (t - L)^{\frac{1}{3}}} H(t - L) \, K\left[\{6(t - L)\}^{-\frac{1}{3}} \int_A^B \frac{N_l \, dl}{(N_l \rho_l)^{\frac{2}{3}}}\right], \tag{138}$$

where $$K(\tau) = \frac{e^{-\frac{1}{12}\pi i}}{2\sqrt{\pi}} \int_0^{\infty} \frac{e^{\pi i} \mathrm{Ai}'(\mu) + Z \, \mathrm{Ai}(\mu)}{e^{\frac{1}{3}\pi i} \mathrm{Ai}'(\mu e^{-\frac{2}{3}\pi i}) + Z \, \mathrm{Ai}(\mu e^{-\frac{2}{3}\pi i})} \mathrm{Ai}(\mu\tau e^{\frac{1}{3}\pi i}) \, d\mu$$
$$+ \frac{e^{-\frac{1}{12}\pi i}}{2\sqrt{\pi}} \int_0^{\infty} \frac{e^{-\frac{1}{3}\pi i} \mathrm{Ai}'(\mu) + Z \, \mathrm{Ai}(\mu)}{e^{\frac{1}{3}\pi i} \mathrm{Ai}'(\mu e^{\frac{2}{3}\pi i}) + Z \, \mathrm{Ai}(\mu e^{\frac{2}{3}\pi i})} \mathrm{Ai}'(\mu\tau e^{-\frac{1}{3}\pi i}) \, d\mu \quad (\tau \geqslant 0)$$
$$= \frac{e^{\frac{1}{4}\pi i}}{2\sqrt{\pi}} \int_0^{\infty} \frac{e^{\pi i} \mathrm{Ai}'(\mu e^{\frac{1}{3}\pi i}) + Z \, \mathrm{Ai}(\mu e^{\frac{1}{3}\pi i})}{e^{\frac{1}{3}\pi i} \mathrm{Ai}'(\mu e^{-\frac{1}{3}\pi i}) + Z \, \mathrm{Ai}(\mu e^{-\frac{1}{3}\pi i})} \mathrm{Ai}(\mu\tau e^{\frac{2}{3}\pi i}) \, d\mu$$
$$+ \frac{e^{-\frac{5}{12}\pi i}}{2\sqrt{\pi}} \int_0^{\infty} \frac{e^{-\frac{1}{3}\pi i} \mathrm{Ai}'(\mu e^{-\frac{1}{3}\pi i}) + Z \, \mathrm{Ai}(\mu e^{-\frac{1}{3}\pi i})}{e^{\frac{1}{3}\pi i} \mathrm{Ai}'(\mu e^{\frac{1}{3}\pi i}) + Z \, \mathrm{Ai}(\mu e^{\frac{1}{3}\pi i})} \mathrm{Ai}(\mu\tau e^{-\frac{2}{3}\pi i}) \, d\mu \quad (\tau \leqslant 0).$$

Note that
$$K(0) = G(0)/(-\tfrac{1}{3})!\,3^{\frac{2}{3}}. \tag{139}$$

An alternative expression for K when $\tau > 0$ is

$$\begin{aligned} K(\tau) &= \frac{e^{-\frac{1}{12}\pi i}}{2\sqrt{\pi}} \int_{\infty \exp(-\frac{2}{3}\pi i)}^{\infty} \frac{e^{\pi i}\,\mathrm{Ai}'(\mu) + Z\,\mathrm{Ai}(\mu)}{e^{\frac{1}{3}\pi i}\,\mathrm{Ai}'(\mu e^{-\frac{2}{3}\pi i}) + Z\,\mathrm{Ai}(\mu e^{-\frac{2}{3}\pi i})}\,\mathrm{Ai}(\mu\tau e^{\frac{1}{3}\pi i})\,d\mu + \frac{e^{-\frac{1}{12}\pi i}}{6\tau\sqrt{\pi}} \\ &= \frac{e^{-\frac{1}{12}\pi i}}{6\tau\sqrt{\pi}} + \sum_s \frac{Z^2 e^{-\frac{3}{4}\pi i}\,\mathrm{Ai}(-\delta_s\tau)}{(\delta_s - Z^2 e^{-\frac{2}{3}\pi i})\,\{\mathrm{Ai}'(\delta_s)\}^2\, 2\sqrt{\pi}}. \end{aligned} \tag{140}$$

When $|\delta_1|\,\tau \gg 1$ the asymptotic formula (41) can be used for the Airy function in the numerator and

$$K(\tau) \sim \frac{e^{-\frac{1}{12}\pi i}}{6\tau\sqrt{\pi}} + \frac{Z^2 \exp\{-\frac{3}{4}\pi i - \frac{2}{3}(-\delta_1\tau)^{\frac{3}{2}}\}}{4\pi(\delta_1 - Z^2 e^{-\frac{2}{3}\pi i})\,(-\delta_1\tau)^{\frac{1}{4}}\,\{\mathrm{Ai}'(\delta_1)\}^2} \tag{141}$$

since only the first term of the series need be retained. Here $\arg(-\delta_1)$ is chosen so that $|\arg(-\delta_1)| < \pi$. Also $Z e^{-\frac{1}{3}\pi i}/\mathrm{Ai}'(\delta_1)$ should be replaced by $-1/\mathrm{Ai}(\delta_1)$ when $Z = 0$.

Similarly, for $\tau < 0$,

$$K(\tau) = \frac{e^{-\frac{5}{12}\pi i}}{2\sqrt{\pi}} \int_{\infty \exp(-\frac{2}{3}\pi i)}^{\infty} \frac{e^{-\frac{1}{3}\pi i}\,\mathrm{Ai}'(\mu e^{-\frac{1}{3}\pi i}) + Z\,\mathrm{Ai}(\mu e^{-\frac{1}{3}\pi i})}{e^{\frac{1}{3}\pi i}\mathrm{Ai}'(\mu e^{\frac{1}{3}\pi i}) + Z\,\mathrm{Ai}(\mu e^{\frac{1}{3}\pi i})}\,\mathrm{Ai}(\mu\tau e^{-\frac{2}{3}\pi i})\,d\mu + \frac{e^{-\frac{1}{12}\pi i}}{6\tau\sqrt{\pi}}.$$

Equation (138) is of course valid only just behind the wave front. Since $L - L_i$ is small compared with L_i (138) can be effectively written

$$\Phi = \frac{\gamma_i}{2^{\frac{3}{2}}(t-L_i)^{\frac{1}{2}}}\mathrm{H}(t-L_i) - \frac{(N_A N_B \rho_A \rho_B)^{\frac{1}{6}}\gamma_A\gamma_B 3^{\frac{2}{3}}\pi^{\frac{1}{2}}}{2^{\frac{1}{3}} e^{-\frac{1}{12}\pi i}(t-L)^{\frac{1}{3}}} K\left[\{6(t-L)\}^{-\frac{1}{3}} \int_A^B \frac{N_l\,dl}{(N_l\rho_l)^{\frac{2}{3}}}\right]\mathrm{H}(t-L). \tag{142}$$

The relative error is, from (101), $O(T^{\frac{1}{6}})$ where T is the time from the passage of a wave front. At the shadow boundary itself (142) predicts

$$\Phi = \frac{\gamma_i}{2^{\frac{3}{2}}(t-L_i)^{\frac{1}{2}}}\mathrm{H}(t-L_i) - \frac{(N_A\rho_A)^{\frac{1}{3}}\gamma_A^2\pi^{\frac{1}{2}}G(0)}{2^{\frac{1}{3}} e^{-\frac{1}{12}\pi i}(t-L_i)^{\frac{1}{3}}(-\frac{1}{3})!}\mathrm{H}(t-L_i)$$

on account of (139). As we move away from the shadow boundary into the illuminated region we obtain the geometrical optics field as and we move away into the shadow region (141) is applicable so that, if we recall (124), we get

$$\Phi = \frac{6^{\frac{3}{4}}\gamma_A\gamma_B(N_A N_B\rho_A\rho_B)^{\frac{1}{6}}}{8\pi^{\frac{1}{2}}(-\delta_1)^{\frac{1}{4}}\{\mathrm{Ai}'(\delta_1)\}^2}\,\frac{Z^2 e^{-\frac{2}{3}\pi i}}{Z^2 e^{-\frac{2}{3}\pi i} - \delta_1}\,\frac{e^{-\zeta}}{(t-L)^{\frac{1}{4}}\left\{\int_A^B \frac{N_l\,dl}{(N_l\rho_l)^{\frac{2}{3}}}\right\}^{\frac{1}{4}}}\mathrm{H}(t-L), \tag{143}$$

where
$$\zeta = \frac{2^{\frac{1}{2}}(-\delta_1)^{\frac{3}{2}}\left\{\int_A^B \frac{N_l\,dl}{(N_l\rho_l)^{\frac{2}{3}}}\right\}^{\frac{3}{2}}}{3^{\frac{3}{2}}(t-L)^{\frac{1}{2}}}.$$

Naturally one must add contributions according to the rule (v) of § 11. Solutions to the problems corresponding to those of §§ 13 to 15 can now be written down. Some calculations, based on a different method, have been made for the circular cylinder and for the horizontally stratified medium for a point of observation in the deep shadow by Friedlander (1958). There is complete agreement between his results and those based on (143).

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The field on the boundary is, from (112),

$$\psi = \left(\frac{\pi}{2s}\right)^{\frac{1}{2}} \gamma_A \,\mathrm{e}^{-sL} F\left\{(\tfrac{1}{2}s)^{\frac{1}{3}}\,\mathrm{e}^{-\frac{1}{6}\pi\mathrm{i}} \int_A^C \frac{N_l\,\mathrm{d}l}{(N_l\rho_l)^{\frac{2}{3}}}\right\}$$

where F is defined by (B 3). Consequently on the boundary

$$\Phi = \frac{\pi^{\frac{1}{2}}\gamma_A 3^{\frac{1}{2}}}{2^{\frac{1}{2}}(t-L)^{\frac{1}{2}}} M\left[\{6(t-L)\}^{-\frac{1}{3}} \int_A^C \frac{N_l\,\mathrm{d}l}{(N_l\rho_l)^{\frac{2}{3}}}\right] \mathrm{H}(t-L),$$

where
$$M(\tau) = \frac{\mathrm{e}^{-\frac{1}{3}\pi\mathrm{i}}}{2\pi\mathrm{i}} \int_{\infty\exp(-\frac{2}{3}\pi\mathrm{i})}^{\infty} \frac{P(\mu\tau\,\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}})}{\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}\mathrm{Ai}'(\mu\,\mathrm{e}^{-\frac{2}{3}\pi\mathrm{i}}) + Z\,\mathrm{Ai}(\mu\,\mathrm{e}^{-\frac{2}{3}\pi\mathrm{i}})}\,\mathrm{d}\mu \quad (\tau \geqslant 0)$$
$$= \frac{\mathrm{e}^{-\frac{1}{3}\pi\mathrm{i}}}{2\pi\mathrm{i}} \int_{\infty\exp(-\frac{2}{3}\pi\mathrm{i})}^{\infty} \frac{P(\mu\tau\,\mathrm{e}^{-\frac{2}{3}\pi\mathrm{i}})}{\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}\mathrm{Ai}'(\mu\,\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}) + Z\,\mathrm{Ai}(\mu\,\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}})}\,\mathrm{d}\mu \quad (\tau \leqslant 0),$$

$$P(\tau) = \frac{1}{2\pi\mathrm{i}} \int_{\infty\exp(-\frac{1}{3}\pi\mathrm{i})}^{\infty\exp(\frac{1}{3}\pi\mathrm{i})} z^{\frac{1}{2}}\,\mathrm{e}^{\frac{1}{3}z^3-\tau z}\,\mathrm{d}z.$$

Note that $$M(0) = F(0)/(3\pi)^{\frac{1}{2}}.$$

At the point of glancing incidence

$$\Phi = \frac{\gamma_A F(0)}{2^{\frac{1}{2}}(t-L)^{\frac{1}{2}}} \mathrm{H}(t-L)$$

which gives, when $Z = 0$,

$$\Phi = \frac{1{\cdot}399\gamma_A}{2^{\frac{1}{2}}(t-L)^{\frac{1}{2}}} \mathrm{H}(t-L)$$

from (B 6).

For $\tau > 0$ we have also

$$M(\tau) = \sum_s \frac{P(-\tau\delta_s)}{(Z^2\,\mathrm{e}^{-\frac{2}{3}\pi\mathrm{i}} - \delta_s)\,\mathrm{Ai}(\delta_s)}.$$

For large τ, $$P(\tau) \sim \frac{\exp(-\frac{2}{3}\tau^{\frac{3}{2}})}{2\pi^{\frac{1}{2}}}$$

so that $$M(\tau) \sim \frac{\exp\{-\frac{2}{3}(-\tau\delta_1)^{\frac{3}{2}}\}}{2\pi^{\frac{1}{2}}(Z^2\,\mathrm{e}^{-\frac{2}{3}\pi\mathrm{i}} - \delta_1)\,\mathrm{Ai}(\delta_1)}.$$

Therefore, well into the shadow,

$$\Phi \sim \frac{3^{\frac{1}{2}}\gamma_A\,\mathrm{e}^{-\zeta}}{2^{\frac{3}{2}}(t-L)^{\frac{1}{2}}(Z^2\,\mathrm{e}^{-\frac{2}{3}\pi\mathrm{i}} - \delta_1)\,\mathrm{Ai}(\delta_1)}.$$

Similarly, on the boundary

$$\frac{\partial\Phi}{\partial n} \sim -N_c\left(\frac{2}{N_c\rho_c}\right)^{\frac{1}{3}} \frac{Z\,\mathrm{e}^{-\frac{1}{6}\pi\mathrm{i}}\pi^{\frac{1}{2}}\gamma_A 3^{\frac{5}{6}}}{2^{\frac{1}{2}}} \frac{\mathrm{d}}{\mathrm{d}t} \frac{\mathrm{H}(t-L)}{(t-L)^{\frac{1}{6}}} M_1\left[\{6(t-L)\}^{-\frac{1}{3}} \int_A^C \frac{N_l\,\mathrm{d}l}{(N_l\rho_l)^{\frac{2}{3}}}\right],$$

where $M_1(\tau)$ is the same as $M(\tau)$ with $P(\tau)$ replaced by $Q(\tau)$ where

$$Q(\tau) = \frac{1}{2\pi\mathrm{i}} \int_{\infty\exp(-\frac{1}{3}\pi\mathrm{i})}^{\infty\exp(\frac{1}{3}\pi\mathrm{i})} z^{-\frac{1}{2}}\,\mathrm{e}^{\frac{1}{3}z^3-\tau z}\,\mathrm{d}z.$$

Observe that $\mathrm{d}Q/\mathrm{d}\tau = -P$.

A word is necessary about the boundary conditions. The substitution $s = \mathrm{i}k$ in (108) gives

$$\frac{\partial\psi}{\partial n} + N_l\left(\frac{2}{N_l\rho_l}\right)^{\frac{1}{3}} s^{\frac{2}{3}} Z\,\mathrm{e}^{-\frac{1}{3}\pi\mathrm{i}}\psi = 0.$$

This may be regarded as the Laplace transform of

$$\int_0^t \frac{\partial}{\partial n}\Phi(u)\,\mathrm{d}u+\frac{3^{\frac{1}{2}}N_l}{\pi}\left(\frac{1}{4N_l\rho_l}\right)^{\frac{1}{3}}Z\,\mathrm{e}^{-\frac{1}{3}\pi\mathrm{i}}\,(-\tfrac{1}{3})!\int_0^t\frac{\Phi(u)}{(t-u)^{\frac{2}{3}}}\,\mathrm{d}u=0$$

or

$$\frac{\partial\Phi}{\partial n}+\frac{3^{\frac{1}{2}}N_l}{\pi}\left(\frac{1}{4N_l\rho_l}\right)^{\frac{1}{3}}Z\,\mathrm{e}^{-\frac{1}{3}\pi\mathrm{i}}\,(-\tfrac{1}{3})!\left\{\Phi(0)\,t^{-\frac{2}{3}}+\int_0^t(t-u)^{-\frac{2}{3}}\frac{\mathrm{d}\Phi(u)}{\mathrm{d}u}\,\mathrm{d}u\right\}=0$$

so that the preceding results apply when this boundary condition is imposed. It reduces to $\partial\Phi/\partial n=0$ and $\Phi=0$ in the cases $Z=0$ and $Z=\infty$. It can be seen that the boundary condition is real only if $Z\,\mathrm{e}^{-\frac{1}{3}\pi\mathrm{i}}$ is real. Since the field must then be real, $K(\tau)$ must be a real multiple of $\mathrm{e}^{-\frac{1}{12}\pi\mathrm{i}}$ (this is supported by (140) because δ_s is real) and both M and M_1 are real.

The boundary condition on Φ could be altered by making Z a function of s (e.g. $Z=s^{\frac{1}{3}}\,\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}$) but then the formulae from (138) onwards would have to be altered because G would no longer involve s only through an exponential factor.

Appendix A

According to (88) the function G is defined by

$$G(\tau)=\frac{\mathrm{e}^{-\frac{1}{12}\pi\mathrm{i}}}{2\pi^{\frac{1}{2}}}\int_0^\infty\frac{\mathrm{e}^{\pi\mathrm{i}}\,\mathrm{Ai}'(\mu)+Z\,\mathrm{Ai}(\mu)}{\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}\,\mathrm{Ai}'(\mu\,\mathrm{e}^{-\frac{2}{3}\pi\mathrm{i}})+Z\,\mathrm{Ai}(\mu\,\mathrm{e}^{-\frac{2}{3}\pi\mathrm{i}})}\,\mathrm{e}^{-\mathrm{i}\mu\tau}\,\mathrm{d}\mu$$
$$+\frac{\mathrm{e}^{-\frac{5}{12}\pi\mathrm{i}}}{2\pi^{\frac{1}{2}}}\int_0^\infty\frac{\mathrm{e}^{-\frac{1}{3}\pi\mathrm{i}}\,\mathrm{Ai}'(\mu\,\mathrm{e}^{-\frac{1}{3}\pi\mathrm{i}})+Z\,\mathrm{Ai}(\mu\,\mathrm{e}^{-\frac{1}{3}\pi\mathrm{i}})}{\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}\,\mathrm{Ai}'(\mu\,\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}})+Z\,\mathrm{Ai}(\mu\,\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}})}\,\mathrm{e}^{\mathrm{i}\mu\tau}\,\mathrm{d}\mu.\qquad(\mathrm{A}\,1)$$

This function, or a constant multiple of it, has been computed by Rice (1954) in the cases $Z=0$ and $Z=\infty$. Values when $Z\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}$ is a real constant are displayed graphically by Wait & Conda (1959). It should be noted that although Wait & Conda define G by (A 3) which is valid only for $\tau>0$ they compute it from the correct formula (A 2).

There are various alternative formulae for G which are useful. It will be assumed in the following that the denominators of the integrands of (A 1) have no zeros in the wedge between the positive real axis and the radius vector from the origin making an angle $\frac{1}{3}\pi$ with the positive real axis. Since these zeros occur at $\delta_s\,\mathrm{e}^{\frac{2}{3}\pi\mathrm{i}}$ and $\delta_s\,\mathrm{e}^{-\frac{1}{3}\pi\mathrm{i}}$ respectively and δ_s is approximately negative real according to the discussion after (52) this assumption can be regarded as justified.

The contour of integration of the second integral in (A 1) can be deformed into the radius vector from the origin making an angle $\frac{1}{3}\pi$ with the positive real axis. This gives

$$G(\tau)=\frac{\mathrm{e}^{-\frac{1}{12}\pi\mathrm{i}}}{2\pi^{\frac{1}{2}}}\int_0^\infty\frac{\mathrm{e}^{\pi\mathrm{i}}\,\mathrm{Ai}'(\mu)+Z\,\mathrm{Ai}(\mu)}{\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}\,\mathrm{Ai}'(\mu\,\mathrm{e}^{-\frac{2}{3}\pi\mathrm{i}})+Z\,\mathrm{Ai}(\mu\,\mathrm{e}^{-\frac{2}{3}\pi\mathrm{i}})}\,\mathrm{e}^{-\mathrm{i}\mu\tau}\,\mathrm{d}\mu$$
$$+\frac{\mathrm{e}^{-\frac{1}{12}\pi\mathrm{i}}}{2\pi^{\frac{1}{2}}}\int_0^\infty\frac{\mathrm{e}^{-\frac{1}{3}\pi\mathrm{i}}\,\mathrm{Ai}'(\mu)+Z\,\mathrm{Ai}(\mu)}{\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}\,\mathrm{Ai}'(\mu\,\mathrm{e}^{\frac{2}{3}\pi\mathrm{i}})+Z\,\mathrm{Ai}(\mu\,\mathrm{e}^{\frac{2}{3}\pi\mathrm{i}})}\,\exp(\mathrm{i}\mu\tau\,\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}})\,\mathrm{d}\mu.\qquad(\mathrm{A}\,2)$$

On the other hand if we deform the contour of the second integral in (A 1) into the radius vector and then change the variable of integration from μ to $-\mu$ we obtain for the second integral

$$\frac{\mathrm{e}^{-\frac{5}{12}\pi\mathrm{i}}}{2\pi^{\frac{1}{2}}}\int_{\infty\exp(-\frac{2}{3}\pi\mathrm{i})}^0\frac{\mathrm{e}^{-\frac{1}{3}\pi\mathrm{i}}\,\mathrm{Ai}'(\mu\,\mathrm{e}^{\frac{2}{3}\pi\mathrm{i}})+Z\,\mathrm{Ai}(\mu\,\mathrm{e}^{\frac{2}{3}\pi\mathrm{i}})}{\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}\,\mathrm{Ai}'(\mu\,\mathrm{e}^{-\frac{2}{3}\pi\mathrm{i}})+Z\,\mathrm{Ai}(\mu\,\mathrm{e}^{-\frac{2}{3}\pi\mathrm{i}})}\,\mathrm{e}^{-\mathrm{i}\mu\tau}\,\mathrm{d}\mu.$$

In the numerator use (47) with $z = \mu\,\mathrm{e}^{-\frac{1}{3}\pi\mathrm{i}}$ and then, if $\tau > 0$,

$$G(\tau) = \frac{\mathrm{e}^{-\frac{1}{12}\pi\mathrm{i}}}{2\pi^{\frac{1}{2}}} \int_{\infty\exp(-\frac{2}{3}\pi\mathrm{i})}^{\infty} \frac{\mathrm{e}^{\pi\mathrm{i}}\,\mathrm{Ai}'(\mu) + Z\,\mathrm{Ai}(\mu)}{\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}\,\mathrm{Ai}'(\mu\,\mathrm{e}^{-\frac{2}{3}\pi\mathrm{i}}) + Z\,\mathrm{Ai}(\mu\,\mathrm{e}^{-\frac{2}{3}\pi\mathrm{i}})}\,\mathrm{e}^{-\mathrm{i}\mu\tau}\,\mathrm{d}\mu + \frac{\mathrm{e}^{-\frac{1}{4}\pi\mathrm{i}}}{2\pi^{\frac{1}{2}}\tau}. \tag{A 3}$$

If $\tau < 0$ leave the second integral of (A 1) alone but deform the contour of integration of the first integral into the radius vector making an angle $\frac{1}{3}\pi$ with the positive real axis. Changing the variable of integration from μ to $-\mu$ and a use of (47) leads to

$$G(\tau) = \frac{\mathrm{e}^{-\frac{5}{12}\pi\mathrm{i}}}{2\pi^{\frac{1}{2}}} \int_{\infty\exp(-\frac{2}{3}\pi\mathrm{i})}^{\infty} \frac{\mathrm{e}^{-\frac{1}{3}\pi\mathrm{i}}\,\mathrm{Ai}'(\mu\,\mathrm{e}^{-\frac{1}{3}\pi\mathrm{i}}) + Z\,\mathrm{Ai}(\mu\,\mathrm{e}^{-\frac{1}{3}\pi\mathrm{i}})}{\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}\,\mathrm{Ai}'(\mu\,\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}) + Z\,\mathrm{Ai}(\mu\,\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}})}\,\mathrm{e}^{\mathrm{i}\mu\tau}\,\mathrm{d}\mu + \frac{\mathrm{e}^{-\frac{1}{4}\pi\mathrm{i}}}{2\pi^{\frac{1}{2}}\tau} \tag{A 4}$$

for $\tau < 0$.

When $\tau > 0$ the contour of (A 3) can be deformed over the poles of the integrand to give

$$G(\tau) = \frac{\mathrm{e}^{-\frac{1}{4}\pi\mathrm{i}}}{2\pi^{\frac{1}{2}}\tau} + \sum_{s=1} \frac{Z^2 \exp(-\frac{1}{12}\pi\mathrm{i} + \mathrm{i}\delta_s\tau\,\mathrm{e}^{-\frac{1}{3}\pi\mathrm{i}})}{(\delta_s\,\mathrm{e}^{\frac{2}{3}\pi\mathrm{i}} - Z^2)\,2\pi^{\frac{1}{2}}\{\mathrm{Ai}'(\delta_s)\}^2} \quad (\tau > 0). \tag{A 5}$$

If $Z = 0$ replace $Z/\mathrm{Ai}'(\delta_s)$ by $-\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}/\mathrm{Ai}(\delta_s)$. This formula is particularly useful when τ is large and positive since then the first term of the series dominates the remainder.

When τ is large and negative, break the path of integration in (A 4) into the two intervals $(\infty\,\mathrm{e}^{-\frac{2}{3}\pi\mathrm{i}}, M)$ and (M, ∞) where M is a large positive real constant but less than $\frac{1}{4}\tau^2$. Integration by parts shows that the interval $(\infty\,\mathrm{e}^{-\frac{2}{3}\pi\mathrm{i}}, M)$ provides

$$\frac{1}{2\mathrm{i}\pi^{\frac{1}{2}}\tau}\left\{\frac{Z+\mathrm{i}M^{\frac{1}{2}}}{Z-\mathrm{i}M^{\frac{1}{2}}} + O\left(\frac{1}{M}\right)\right\} \exp(-\tfrac{1}{4}\pi\mathrm{i} + \tfrac{4}{3}\mathrm{i}M^{\frac{3}{2}} + \mathrm{i}M\tau) + O\left(\frac{1}{\tau^2}\right) \tag{A 6}$$

since the asymptotic expansions of the Airy functions can be employed for $\mu \geqslant M$.

The interval (M, ∞) gives

$$\frac{\mathrm{e}^{-\frac{1}{4}\pi\mathrm{i}}}{2\pi^{\frac{1}{2}}} \int_M^{\infty} \left\{\frac{Z+\mathrm{i}\mu^{\frac{1}{2}}}{Z-\mathrm{i}\mu^{\frac{1}{2}}} + O\left(\frac{1}{\mu}\right)\right\} \exp(\tfrac{4}{3}\mathrm{i}\mu^{\frac{3}{2}} + \mathrm{i}\mu\tau)\,\mathrm{d}\mu.$$

There is a point of stationary phase at $\mu^{\frac{1}{2}} = -\frac{1}{2}\tau$. However, a more elaborate treatment than the usual one is required since an estimate of the next term is needed. Put $\mu^{\frac{1}{2}} = u\tau$ so that the integral becomes

$$\frac{\tau^2\,\mathrm{e}^{-\frac{1}{4}\pi\mathrm{i}}}{\pi^{\frac{1}{2}}} \int_{-M^{\frac{1}{2}}/\tau}^{\infty} \left\{\frac{Z-\mathrm{i}u\tau}{Z+\mathrm{i}u\tau} + O\left(\frac{1}{u^2\tau^2}\right)\right\} \exp\{\mathrm{i}\tau^3(u^2 - \tfrac{4}{3}u^3)\}\,u\,\mathrm{d}u.$$

The transformation $u^2 - \frac{4}{3}u^3 = \frac{1}{12} - v$ gives

$$\frac{\tau^2 \exp(-\frac{1}{4}\pi\mathrm{i} + \frac{1}{12}\mathrm{i}\tau^3)}{4\pi^{\frac{1}{2}}} \Bigg[\int_0^{\infty} \left\{\frac{Z-\mathrm{i}u\tau}{Z+\mathrm{i}u\tau} + O\left(\frac{1}{u^2\tau^2}\right)\right\} \frac{\mathrm{e}^{-\mathrm{i}v\tau^3}}{u-\frac{1}{2}}\,\mathrm{d}v$$
$$- \int_0^{\frac{1}{12}-\frac{4}{3}(M^{\frac{3}{2}}/\tau^3)-M/\tau^2} \left\{\frac{Z-\mathrm{i}u\tau}{Z+\mathrm{i}u\tau} + O\left(\frac{1}{u^2\tau^2}\right)\right\} \frac{\mathrm{e}^{-\mathrm{i}v\tau^3}}{u-\frac{1}{2}}\,\mathrm{d}v\Bigg],$$

where in the first integral $u - \frac{1}{2} \approx v^{\frac{1}{2}}(1 - \frac{2}{3}v^{\frac{1}{2}})$ when $u \approx \frac{1}{2}$ and in the second integral $u - \frac{1}{2} \approx -v^{\frac{1}{2}}(1 + \frac{2}{3}v^{\frac{1}{2}})$ when $u \approx \frac{1}{2}$. The standard theory of the asymptotic behaviour of a Fourier integral now supplies

$$\tfrac{1}{2}(-\tau)^{\frac{1}{2}} \frac{Z-\frac{1}{2}\mathrm{i}\tau}{Z+\frac{1}{2}\mathrm{i}\tau}\,\mathrm{e}^{\frac{1}{12}\mathrm{i}\tau^3}\left\{1 + O\left(\frac{1}{\tau^2}\right)\right\} + \frac{Z+\mathrm{i}M^{\frac{1}{2}}}{Z-\mathrm{i}M^{\frac{1}{2}}}\,\frac{\mathrm{i}\exp(-\frac{1}{4}\pi\mathrm{i} + \frac{4}{3}\mathrm{i}M^{\frac{3}{2}} + \mathrm{i}M\tau)}{2\tau\pi^{\frac{1}{2}}(1 + 2M^{\frac{1}{2}}/\tau)} + O\left(\frac{1}{\tau^4}\right).$$

Combining this with (A 6) we see that as $\tau \to -\infty$

$$G(\tau) = \tfrac{1}{2}(-\tau)^{\frac{1}{2}} \frac{Z-\frac{1}{2}\mathrm{i}\tau}{Z+\frac{1}{2}\mathrm{i}\tau} \mathrm{e}^{\frac{1}{12}\mathrm{i}\tau^3} + \frac{\mathrm{e}^{-\frac{1}{4}\pi\mathrm{i}}}{2\pi^{\frac{1}{2}}\tau} + O(\tau^{-\frac{3}{2}}). \tag{A 7}$$

With regard to $G''(\tau)$ we have

$$G''(\tau) = \frac{\mathrm{e}^{-\frac{1}{4}\pi\mathrm{i}}}{\pi^{\frac{1}{2}}\tau^3} - \sum_{s=1} \frac{\delta_s^2 Z^2 \exp\left(-\frac{3}{4}\pi\mathrm{i} + \mathrm{i}\delta_s \tau\, \mathrm{e}^{-\frac{1}{3}\pi\mathrm{i}}\right)}{2\pi^{\frac{1}{2}}(\delta_s\, \mathrm{e}^{\frac{2}{3}\pi\mathrm{i}} - Z^2)\{\mathrm{Ai}'(\delta_s)\}^2} \quad (\tau > 0). \tag{A 8}$$

For $\tau < 0$ note that the first term of (A 7) comes from the point of stationary phase so that if its contribution is omitted

$$G''(\tau) \sim \frac{\mathrm{e}^{-\frac{1}{4}\pi\mathrm{i}}}{\pi^{\frac{1}{2}}\tau^3} \tag{A 9}$$

as $\tau \to -\infty$.

When $Z = 0$

$$G(0) = -0{\cdot}308\mathrm{e}^{-\frac{1}{12}\pi\mathrm{i}} = -0{\cdot}298 + \mathrm{i}0{\cdot}0798 \tag{A 10}$$

and, when $Z = \infty$,

$$G(0) = 0{\cdot}354\mathrm{e}^{-\frac{1}{12}\pi\mathrm{i}} = 0{\cdot}342 - \mathrm{i}0{\cdot}0917. \tag{A 11}$$

Values of $G(0)$ when $Z\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}}$ is a real constant are displayed graphically in Wait & Conda (1959).

Appendix B

The function F is defined in §12, namely

$$F(\tau) = \frac{\mathrm{e}^{-\frac{1}{3}\pi\mathrm{i}}}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}\mu\tau}}{\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}\,\mathrm{Ai}'(\mu\,\mathrm{e}^{-\frac{2}{3}\pi\mathrm{i}}) + Z\,\mathrm{Ai}(\mu\,\mathrm{e}^{-\frac{2}{3}\pi\mathrm{i}})}\,\mathrm{d}\mu. \tag{B 1}$$

By replacing μ by $\mu\,\mathrm{e}^{\frac{2}{3}\pi\mathrm{i}}$ we obtain

$$F(\tau) = \frac{\mathrm{e}^{-\frac{2}{3}\pi\mathrm{i}}}{2\pi\mathrm{i}} \int_{\infty\exp(-\frac{2}{3}\pi\mathrm{i})}^{\infty\exp(\frac{2}{3}\pi\mathrm{i})} \frac{\exp(-\mathrm{i}\mu\tau\,\mathrm{e}^{\frac{2}{3}\pi\mathrm{i}})}{\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}\,\mathrm{Ai}'(\mu) + Z\,\mathrm{Ai}(\mu)}\,\mathrm{d}\mu. \tag{B 2}$$

On the other hand deformation of the contour in (B 1) gives

$$F(\tau) = \frac{\mathrm{e}^{-\frac{1}{3}\pi\mathrm{i}}}{2\pi\mathrm{i}} \int_{\infty\exp(-\frac{2}{3}\pi\mathrm{i})}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}\mu\tau}}{\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}\,\mathrm{Ai}'(\mu\,\mathrm{e}^{-\frac{2}{3}\pi\mathrm{i}}) + Z\,\mathrm{Ai}(\mu\,\mathrm{e}^{-\frac{2}{3}\pi\mathrm{i}})}\,\mathrm{d}\mu. \tag{B 3}$$

For $\tau > 0$ the contour of integration can be deformed over the poles in the lower half plane to give

$$F(\tau) = \sum_{s=1} \frac{\exp(\mathrm{i}\delta_s\tau\,\mathrm{e}^{-\frac{1}{3}\pi\mathrm{i}})}{(Z^2\,\mathrm{e}^{-\frac{2}{3}\pi\mathrm{i}} - \delta_s)\,\mathrm{Ai}(\delta_s)} \quad (\tau > 0). \tag{B 4}$$

If Z is infinite replace $Z\mathrm{e}^{-\frac{1}{3}\pi\mathrm{i}}\,\mathrm{Ai}(\delta_s)$ by $-\mathrm{Ai}'(\delta_s)$.

When $\tau \to -\infty$ the same method as was used for G in appendix A can be adopted to supply

$$F(\tau) \sim \frac{2\tau\,\mathrm{e}^{\frac{1}{3}\mathrm{i}\tau^3}}{\tau - \mathrm{i}Z} \tag{B 5}$$

as $\tau \to -\infty$.

Computations of F (in some cases multiplied by a constant factor) have been made by Rice (1954) ($Z = \infty$), Fock (1946) ($Z = 0$) and Wait & Conda (1958) (Z of the form $-\mathrm{i}\rho^{\frac{5}{6}}(\alpha\rho - \mathrm{i})^{-\frac{1}{2}}$ for various real values of α and ρ). It is worth noting that

$$F(0) = 1{\cdot}399 \qquad (Z = 0), \tag{B 6}$$

$$ZF(0) = -0{\cdot}77\,\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}} \quad (Z = \infty). \tag{B 7}$$

References

Beckmann, P. & Franz, W. 1957 *Z. Naturforsch.* **12** *a*, 533.
Brekhovskikh, L. 1960 *Waves in layered media.* New York: Academic Press.
Bremmer, H. 1949 *Terrestrial radio waves.* New York: Elsevier.
Clemmow, P. C. 1959 *Univ. of Michigan Studies in Radar Cross Sections* XXXII.
Erdélyi, A. 1955 *Asymptotic expansions*, O.N.R. Tech. Rep. 3.
Felsen, L. B. 1959 *Brooklyn Polytechnic Microwave Res. Inst. Memo.* 55.
Fock, V. 1945 *J. Phys. U.S.S.R.* **9**, 255.
Fock, V. 1946 *J. Phys. U.S.S.R.* **10**, 130, 399.
Fock, V. 1948 *Phil. Mag.* (7), **39**, 149.
Fock, V. 1951 *Uspekhi Fis. Nauk*, **43**, 587.
Franz, W. 1954 *Z. Naturforsch.* **9** *a*, 705.
Franz, W. & Beckmann, P. 1956 *Trans. Inst. Radio Engrs.* AP-4, 203.
Franz, W. & Depperman, K. 1952 *Ann. Phys., Lpz.*, **10**, 361.
Friedlander, F. G. 1955 *New York Univ. Res. Rep.* EM–76.
Friedlander, F. G. 1958 *Sound pulses.* Cambridge University Press.
Goodrich, R. F. 1959 *Trans. Inst. Radio Engrs.* AP–7, 528.
Goriainov, A. S. 1958 *Radiotekh. Elektron*, **3**, 603.
Jeffreys, H. & Lapwood, E. R. 1957 *Proc. Roy. Soc.* A, **241**, 455.
Jones, D. S. 1955 *Phil. Mag.* (7), **46**, 957.
Jones, D. S. 1957 *a* *Proc. Roy. Soc.* A, **239**, 338.
Jones, D. S. 1957 *b* *Proc. Roy. Soc.* A, **240**, 206.
Jones, D. S. 1962 *The theory of electromagnetism.* London: Pergamon Press.
Jones, D. S. & Kline, M. 1958 *J. Math. Phys.* **37**, 1.
Jones, D. S. & Whitham, G. B. 1957 *Proc. Camb. Phil. Soc.* **53**, 691.
Kear, G. 1956 *New York Univ. Res. Rep.* EM–86.
Keller, J. B. 1956 *Trans. Inst. Radio Engrs.* AP–4, 312.
Levy, B. 1958 *New York Univ. Res. Rep.* EM–121.
Levy, B. & Keller, J. B. 1957 *New York Univ. Res. Rep.* EM–109.
Macdonald, H. M. 1913 *Phil. Trans.* A, **212**, 299.
Marcuvitz, N. 1951 *Commun. Pur. Appl. Math.* **4**, 263.
Rice, S. O. 1954 *Bell System Techn. J.* **33**, 417.
Seckler, B. D. & Keller, J. B. 1959 *J. Acoust. Soc. Amer.* **31**, 192, 206.
Ursell, F. J. 1957 *Proc. Camb. Phil. Soc.* **53**, 115.
Van der Pol, B. & Bremmer, H. 1937 *Phil. Mag.* (7), **24**, 141, 825.
Van der Pol, B. & Bremmer, H. 1938 *Phil. Mag.* (7), **25**, 817.
Van der Pol, B. & Bremmer, H. 1939 *Phil. Mag.* (7), **27**, 261.
Wait, J. R. & Conda, A. M. 1958 *Trans. Inst. Radio Engrs.* AP–6, 1957.
Wait, J. R. & Conda, A. M. 1959 *J. Res. Nat. Bur. Stand.* **63** D, 181.
White, F. P. 1922 *Proc. Roy. Soc.* A, **100**, 505.
Wu, T. T. 1956 *Phys. Rev.* **104**, 1201.